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EFFECTIVE CUBATURE FORMULA FOR THE APPROXIMATE CALCULATION OF THE TRIPLE INTEGRAL ON THE CLASS OF DIFFERENTIABLE FUNCTIONS

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Abstract

The rapid development of digital technologies encourages scientists to create new or improve existing algorithms for solving technical problems. It is time to develop mathematical models with different types of input or experimental data. In particular, in the digital signal and image processing, algorithms for the approximate calculation of integrals from rapidly oscillating functions of many variables are used in the case when the input information is given by the values of the function on lines or planes. The specified results can be easily transferred to the problem of numerical integration of functions of many variables and find their application in mathematical modeling of technical processes. The purpose of this work is to present an effective cubature formula for the approximate calculation of the triple integral on the class of differentiable functions. We consider the formula as effective if it uses less function values compared to the classic algorithm.

Keywords: mathematical modeling, numerical integration of the function of three variables, cubature formula.

INTRODUCTION

Many scientific works have been devoted to the issue of numerical integration of several variables. The most famous classical methods are the method of central rectangles, the method of trapezoids, Simpson's formula and Gauss's formula. In the multidimensional case the classification of works can be done by the type of information about function. Cubic formulas for the approximate calculation of multiple integrals can use information about a function not only as function values at points. Currently, the solution of technical problems requires cubature formulas for the approximate calculation of integrals in cases where information about a function is given by its values on planes and lines. It is possible to construct such cubature formulas using new information operators [1, 2]. For example, in digital processing of signals and images, such an approach is implemented.

Integration of rapidly oscillating functions of one and many variables is one of the important tasks in digital signal and image processing. Many studies are devoted to this topic [3-6]. Numerical integration of the approximate calculation of integrals from rapidly oscillating functions using new information operators made it possible to construct cubature formulas using different types of information specification. Constructed cubature formulas use the function values on planes, lines, and points as data [7-11].

The work [12] gives cubature formulas for the approximate calculation of the double integral, in the case when information about the function is given on lines. The issue of optimal selection of lines is considered. Formulas have high calculation accuracy. In addition, this paper presents an effective (in terms of the number of function values used) cubature formula.

Works [13, 14] demonstrate cubature formulas for the approximate calculation of triple integrals, which use values of functions on planes and straight lines. The purpose of this work is to present the cubic formula for the approximate calculation of the triple integral, which uses the methods implemented in [2, 13, 14]. The proposed cubature formula uses fewer function values, compared to classical methods, to achieve the given accuracy.

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The cubature formula is built on economic schemes of spline interpolation of the function of three variables using new information operators [2, pp. 213-220].

For an approximate calculation of the integral from functions of three variables of the form

$$I(f)=\int_0^1\int_0^1\int_0^1 f(x,y,z)dxdydz$$

a cubature formula is proposed

$$I(f)\approx\Phi(f)=$$

$$=\frac{1}{\ell\cdot\ell^{3/2}\cdot\ell^{3}}\sum_{k=1}^{\ell}\sum_{\tilde{j}=1}^{\ell^{3/2}}\sum_{\overline{s}=1}^{\ell^{3}} f(x_k,\tilde{y}_{\tilde{j}},\overline{z}_{\overline{s}})+$$

$$+\frac{1}{\ell\cdot\ell^{3}\cdot\ell^{3/2}}\sum_{k=1}^{\ell}\sum_{\overline{j}=1}^{\ell^{3}}\sum_{\tilde{s}=1}^{\ell^{3/2}} f\left(x_k,\overline{y}_{\overline{j}},\tilde{z}_{\tilde{s}}\right)-$$

$$-\frac{1}{\ell\cdot\ell^{3/2}\cdot\ell^{3/2}}\sum_{k=1}^{\ell}\sum_{\tilde{j}=1}^{\ell^{3/2}}\sum_{\tilde{s}=1}^{\ell^{3/2}} f(x_k,\tilde{y}_{\tilde{j}},\tilde{z}_{\tilde{s}})+$$

$$+\frac{1}{\ell^{3}\cdot\ell\cdot\ell^{3/2}}\sum_{\overline{k}=1}^{\ell^{3}}\sum_{j=1}^{\ell}\sum_{\tilde{s}=1}^{\ell^{3/2}} f\left(\overline{x}_{\overline{k}},y_j,\tilde{z}_{\tilde{s}}\right)+$$

$$+\frac{1}{\ell^{3/2}\cdot\ell\cdot\ell^{3}}\sum_{\tilde{k}=1}^{\ell^{3/2}}\sum_{j=1}^{\ell}\sum_{\overline{s}=1}^{\ell^{3}} f\left(\tilde{x}_{\tilde{k}},y_j,\overline{z}_{\overline{s}}\right)-$$

$$-\frac{1}{\ell^{3/2}\cdot\ell\cdot\ell^{3/2}}\sum_{\tilde{k}=1}^{\ell^{3/2}}\sum_{j=1}^{\ell}\sum_{\tilde{s}=1}^{\ell^{3/2}} f(\tilde{x}_{\tilde{k}},y_j,\tilde{z}_{\tilde{s}})+$$

$$+\frac{1}{\ell\cdot\ell^{3/2}\cdot\ell^{3}}\sum_{s=1}^{\ell}\sum_{\tilde{k}=1}^{\ell^{3/2}}\sum_{\overline{j}=1}^{\ell^{3}} f\left(x_k,\overline{y}_{\overline{j}},\tilde{z}_{\tilde{s}}\right)+$$

$$+\frac{1}{\ell^{3}\cdot\ell^{3/2}\cdot\ell}\sum_{\overline{k}=1}^{\ell^{3}}\sum_{\tilde{j}=1}^{\ell^{3/2}}\sum_{s=1}^{\ell} f\left(\overline{x}_{\overline{k}},y_j,\tilde{z}_{\tilde{s}}\right)-$$

$$-\frac{1}{\ell^{3/2}\cdot\ell^{3/2}\cdot\ell}\sum_{\tilde{k}=1}^{\ell^{3/2}}\sum_{\tilde{j}=1}^{\ell^{3/2}}\sum_{s=1}^{\ell} f(\tilde{x}_{\tilde{k}},\tilde{y}_{\tilde{j}},z_s)-$$

$$-\frac{1}{\ell\cdot\ell\cdot\ell^{3}}\sum_{k=1}^{\ell}\sum_{j=1}^{\ell}\sum_{\overline{s}=1}^{\ell^{3}} f(x_k,y_j,\overline{z}_{\overline{s}})-$$

$$-\frac{1}{\ell\cdot\ell^{3}\cdot\ell}\sum_{k=1}^{\ell}\sum_{\overline{j}=1}^{\ell^{3}}\sum_{s=1}^{\ell} f(x_k,\overline{y}_{\overline{j}},z_s)-$$

$$-\frac{1}{\ell^{3}\cdot\ell\cdot\ell}\sum_{\overline{k}=1}^{\ell^{3}}\sum_{j=1}^{\ell}\sum_{s=1}^{\ell} f(\overline{x}_{\overline{k}},y_j,z_s)+$$

$$+\frac{1}{\ell\cdot\ell\cdot\ell}\sum_{k=1}^{\ell}\sum_{j=1}^{\ell}\sum_{s=1}^{\ell} f(x_k,y_j,z_s),$$

$$x_k=k\Delta-\frac{\Delta}{2},\ y_j=j\Delta-\frac{\Delta}{2},$$

$$z_s=s\Delta-\frac{\Delta}{2},\quad \Delta=\frac{1}{\ell},\ k,j,s=\overline{1,\ell},$$

$$\tilde{x}_{\tilde{k}}=\tilde{k}\Delta_1-\frac{\Delta_1}{2},\ \tilde{y}_{\tilde{j}}=\tilde{j}\Delta_1-\frac{\Delta_1}{2},$$

$$\tilde{z}_{\tilde{s}}=\tilde{s}\Delta_1-\frac{\Delta_1}{2},\quad \Delta_1=\frac{1}{\ell^{3/2}},\quad \tilde{k},\tilde{j},\tilde{s}=\overline{1,\ell^{3/2}}$$

,

$$\overline{x}_{\overline{k}}=\overline{k}\Delta_2-\frac{\Delta_2}{2},\ \overline{y}_{\overline{j}}=\overline{j}\Delta_2-\frac{\Delta_2}{2},$$

$$.\ \overline{z}_{\overline{s}}=\overline{s}\Delta_2-\frac{\Delta_2}{2},\quad \Delta_2=\frac{1}{\ell^{3}},\quad \overline{k},\overline{j},\overline{s}=\overline{1,\ell^{3}}$$

.

Theorem. If

$$\left|f^{(0,0,1)}(x,y,z)\right|\le M,$$

$$\left|f^{(1,1,0)}(x,y,z)\right|\le\overline{M},\quad \left|f^{(1,0,1)}(x,y,z)\right|\le\overline{M},$$

$$\left|f^{(0,1,1)}(x,y,z)\right| \le \overline{M}, \quad \left|f^{(1,1,1)}(x,y,z)\right| \le \tilde{M},$$

then

$$\left|I(f)-\Phi(f)\right| \le \left(\frac{\tilde{M}}{64}+\frac{3\overline{M}}{16}+\frac{9}{4}M\right)\frac{1}{\ell^3}.$$

Proof. Let define operators

$$J_1 f(x,y,z)=\sum_{k=1}^{\ell} f(x_k,y,z)h_{1k}^0(x),$$

$$J_2 f(x,y,z)=\sum_{j=1}^{\ell} f(x,y_j,z)h_{2j}^0(y),$$

$$J_3 f(x,y,z)=\sum_{s=1}^{\ell} f(x,y,z_s)h_{3s}^0(z),$$

$$\tilde{J}_1 f(x,y,z)=\sum_{\tilde{k}=1}^{\ell^{3/2}} f(\tilde{x}_{\tilde{k}},y,z)\tilde{h}_{1\tilde{k}}^0(x),,$$

$$\tilde{J}_2 f(x,y,z)=\sum_{\tilde{j}=1}^{\ell^{3/2}} f(x,\tilde{y}_{\tilde{j}},z)\tilde{h}_{2\tilde{j}}^0(y),,$$

$$\tilde{J}_3 f(x,y,z)=\sum_{\tilde{s}=1}^{\ell^{3/2}} f(x,y,\tilde{z}_{\tilde{s}})\tilde{h}_{3\tilde{s}}^0(z),$$

$$X_k=\left[x_{k-1/2},x_{k+1/2}\right],\ Y_j=\left[y_{j-1/2},y_{j+1/2}\right],$$

$$Z_s=\left[z_{s-1/2},z_{s+1/2}\right],\ \tilde{X}_{\tilde{k}}=\left[\tilde{x}_{\tilde{k}-1/2},\tilde{x}_{\tilde{k}+1/2}\right],$$

$$\tilde{Y}_{\tilde{j}}=\left[\tilde{y}_{\tilde{j}-1/2},\tilde{y}_{\tilde{j}+1/2}\right],\quad \tilde{Z}_{\tilde{s}}=\left[\tilde{z}_{\tilde{s}-1/2},\tilde{z}_{\tilde{s}+1/2}\right],$$

$$\overline{X}_{\overline{k}}=\left[\overline{x}_{\overline{k}-1/2},\overline{x}_{\overline{k}+1/2}\right],\ \overline{Y}_{\overline{j}}=\left[\overline{y}_{\overline{j}-1/2},\overline{y}_{\overline{j}+1/2}\right],$$

$$\overline{Z}_{\overline{s}}=\left[\overline{z}_{\overline{s}-1/2},\overline{z}_{\overline{s}+1/2}\right]$$

$$h_{1k}^0(x)=\begin{cases}1, x\in X_k,\\ 0, x\notin X_k,\end{cases}\quad h_{2j}^0(y)=\begin{cases}1, y\in Y_j,\\ 0, y\notin Y_j,\end{cases}$$

$$h_{3s}^0(z)=\begin{cases}1, z\in Z_s,\\ 0, z\notin Z_s,\end{cases},$$

$$\tilde{h}_{1\tilde{k}}^0(x)=\begin{cases}1, x\in \tilde{X}_{\tilde{k}},\\ 0, x\notin \tilde{X}_{\tilde{k}},\end{cases}\quad \tilde{h}_{2\tilde{j}}^0(y)=\begin{cases}1, y\in \tilde{Y}_{\tilde{j}},\\ 0, y\notin \tilde{Y}_{\tilde{j}},\end{cases}$$

$$\tilde{h}_{3\tilde{s}}^0(z)=\begin{cases}1, z\in \tilde{Z}_{\tilde{s}},\\ 0, z\notin \tilde{Z}_{\tilde{s}}.\end{cases}$$

$$\overline{h}_{1\overline{k}}^0(x)=\begin{cases}1, x\in \overline{X}_{\overline{k}},\\ 0, x\notin \overline{X}_{\overline{k}},\end{cases}\quad \overline{h}_{2\overline{j}}^0(y)=\begin{cases}1, y\in \overline{Y}_{\overline{j}},\\ 0, y\notin \overline{Y}_{\overline{j}},\end{cases}$$

$$\overline{h}_{3\overline{s}}^0(z)=\begin{cases}1, z\in \overline{Z}_{\overline{s}},\\ 0, z\notin \overline{Z}_{\overline{s}}.\end{cases}$$

The work [13, 14] deals with operators

$$\begin{aligned} Jf(x,y,z)&=J_1 f(x,y,z)+J_2 f(x,y,z)+\\ &+J_3 f(x,y,z)-\\ &-J_1J_2 f(x,y,z)-J_2J_3 f(x,y,z)-\\ &-J_1J_3 f(x,y,z)+J_1J_2J_3 f(x,y,z)\end{aligned}$$

and

$$\begin{aligned} \tilde{J} f(x,y,z)&=J_1\tilde{J}_2 f(x,y,z)+J_1\tilde{J}_3 f(x,y,z)-\\ &-J_1\tilde{J}_2\tilde{J}_3 f(x,y,z)+J_2\tilde{J}_1 f(x,y,z)+\\ &+J_2\tilde{J}_3 f(x,y,z)-J_2\tilde{J}_1\tilde{J}_3 f(x,y,z)+J_3\tilde{J}_1 f(x,y,z)+\\ &+J_3\tilde{J}_2 f(x,y,z)-J_3\tilde{J}_1\tilde{J}_2 f(x,y,z)-\\ &-J_1J_2 f(x,y,z)-J_1J_3 f(x,y,z)-\\ &-J_2J_3 f(x,y,z)+J_1J_2J_3 f(x,y,z)\end{aligned}$$

and cubature formula

$$\tilde{\Phi}(f)=\int_0^1\int_0^1\int_0^1 \tilde{J} f(x,y,z)\,dxdydz.$$

It was proven [14] that

$$\left|I(f)-\tilde{\Phi}(f)\right| \le \left(\frac{\tilde{M}}{64}+\frac{3\overline{M}}{16}\right)\frac{1}{\ell^3}.$$

Let consider operator

$$\begin{aligned} \overline{J} f(x,y,z)&=J_1\tilde{J}_2\overline{J}_3 f(x,y,z)+\\ &+J_1\tilde{J}_3\overline{J}_2 f(x,y,z)-J_1\tilde{J}_2\tilde{J}_3 f(x,y,z)+\\ &+J_2\tilde{J}_1\overline{J}_3 f(x,y,z)+J_2\tilde{J}_3\overline{J}_1 f(x,y,z)-\\ &-J_2\tilde{J}_1\tilde{J}_3 f(x,y,z)+J_3\tilde{J}_1\overline{J}_2 f(x,y,z)+\\ &+J_3\tilde{J}_2\overline{J}_1 f(x,y,z)-J_3\tilde{J}_1\tilde{J}_2 f(x,y,z)-\\ &-J_1J_2\overline{J}_3 f(x,y,z)-J_1J_3\overline{J}_2 f(x,y,z)-\\ &-J_2J_3\overline{J}_1 f(x,y,z)+J_1J_2J_3 f(x,y,z).\end{aligned}$$

Cubature formula $\Phi(f)$ can be represented as

$$\Phi(f)=\int_0^1\int_0^1\int_0^1 \overline{J} f(x,y,z)dxdydz .$$

Then

$$\left|I(f)-\Phi(f)\right|=\left|I(f)-\tilde{\Phi}(f)+\tilde{\Phi}(f)-\Phi(f)\right|\le$$

$$\le\left|I(f)-\tilde{\Phi}(f)\right|+\left|\tilde{\Phi}(f)-\Phi(f)\right|\le$$

$$\le\left(\frac{\tilde{M}}{64}+\frac{3\overline{M}}{16}\right)\frac{1}{\ell^3}+\left|\tilde{\Phi}(f)-\Phi(f)\right|.$$

Thus, to prove the theorem, it is necessary to obtain the evaluation of the expression

$$\left|\tilde{\Phi}(f)-\Phi(f)\right|\le$$

$$\le\int_0^1\int_0^1\int_0^1\left|\tilde{J}f(x,y,z)-\overline{J}f(x,y,z)\right|dxdydz\le$$

$$\le\int_0^1\int_0^1\int_0^1\left|\left(J_1\tilde{J}_2-J_1\tilde{J}_2\overline{J}_3\right)f(x,y,z)\right|dxdydz+$$

$$+\int_0^1\int_0^1\int_0^1\left|\left(J_1\tilde{J}_3+J_1\tilde{J}_3\overline{J}_2\right)f(x,y,z)\right|dxdydz+$$

$$+\int_0^1\int_0^1\int_0^1\left|\left(J_2\tilde{J}_1-J_2\tilde{J}_1\overline{J}_3\right)f(x,y,z)\right|dxdydz+$$

$$+\int_0^1\int_0^1\int_0^1\left|\left(J_2\tilde{J}_3-J_2\tilde{J}_3\overline{J}_1\right)f(x,y,z)\right|dxdydz+$$

$$+\int_0^1\int_0^1\int_0^1\left|\left(J_3\tilde{J}_1-J_3\tilde{J}_1\overline{J}_2\right)f(x,y,z)\right|dxdydz+$$

$$+\int_0^1\int_0^1\int_0^1\left|\left(J_3\tilde{J}_2-J_3\tilde{J}_2\overline{J}_1\right)f(x,y,z)\right|dxdydz+$$

$$+\int_0^1\int_0^1\int_0^1\left|\left(J_1J_2-J_1J_2\overline{J}_3\right)f(x,y,z)\right|dxdydz\ +$$

$$+\int_0^1\int_0^1\int_0^1\left|\left(J_1J_3-J_1J_3\overline{J}_2\right)f(x,y,z)\right|dxdydz+$$

$$+\int_0^1\int_0^1\int_0^1\left|\left(J_2J_3-J_2J_3\overline{J}_1\right)f(x,y,z)\right|dxdydz\le$$

$$\le 3M\ell\Delta\ell^{3/2}\Delta_1\ell^3\frac{\Delta_2^2}{4}+$$

$$3M\ell\Delta\ell^{3/2}\Delta_1\ell^3\frac{\Delta_2^2}{4}+$$

$$+3M\ell\Delta\ell^{3/2}\Delta_1\ell^3\frac{\Delta_2^2}{4}=$$

$$=\frac{3}{4}M\frac{1}{\ell^3}+\frac{3}{4}M\frac{1}{\ell^3}+\frac{3}{4}M\frac{1}{\ell^3}=\frac{9}{4}M\frac{1}{\ell^3}.$$

By combining the estimates, we get

$$\left|I(f)-\Phi(f)\right|\le\left(\frac{\tilde{M}}{64}+\frac{3\overline{M}}{16}+\frac{9}{4}M\right)\frac{1}{\ell^3}.$$

The theorem is proved.

It is proposed to compare the formula $\Phi(f)$ with the classical formula, in which the number of function values is ℓ^9:

$$\Phi cl(f)=\frac{1}{\ell^3\cdot\ell^3\cdot\ell^3}\sum_{\overline{k}=1}^{\ell^3}\sum_{\overline{j}=1}^{\ell^3}\sum_{\overline{s}=1}^{\ell^3} f(\overline{x}_{\overline{k}},\overline{y}_{\overline{j}},\overline{z}_{\overline{s}})$$

.

In the case that

$$\left|f^{(1,0,0)}(x,y,z)\right|\le M,\quad \left|f^{(0,1,0)}(x,y,z)\right|\le M,$$

$$\left|f^{(0,0,1)}(x,y,z)\right|\le M\ ,$$

it can be proved that

$$\left|I(f)-\Phi cl(f)\right|\le\frac{3M}{4\ell^3}.$$

NUMERICAL EXPERIMENT

Let the $N=\ell^9$ values of the function be given $f(x,y,z)$. The numerical experiment aims to demonstrate that the cubature formula $\Phi cl(f)$ for the approximate calculation and achieving the error $O\left(\frac{1}{\sqrt[3]{N}}\right)=O\left(\frac{1}{\ell^3}\right)$ uses all values $N=\ell^9$ of the function $f(x,y,z)$.

The cubature formula $\Phi(f)$ uses $O\left(\sqrt[18]{N^7}\right)=O\left(\sqrt{\ell^7}\right)$ times fewer function values $f(x,y,z)$ to achieve the same error $O\left(\frac{1}{\sqrt[3]{N}}\right)=O\left(\frac{1}{\ell^3}\right)$.

Let consider the function $f(x,y,z)=\sin(x+y+z)$, then the exact value of the integral $I(f)=0,879354930645401$.

Table 1 shows the numerical results of the approximate calculation $I(f)$ using the cubature formula $\Phi(f)$, the approximation error $\varepsilon_1=\left|I(f)-\Phi(I)\right|$ and the number of used function values Q.

Table. 1. Calculation $I(f)$ according to the formula $\Phi(f)$

ℓ	$\sqrt{\ell^3}$	ℓ^3	ε_1	$Q=\ell^{11/2}$
4	8	64	$2.57\cdot 10^{-5}$	2048
9	27	729	$1.98\cdot 10^{-7}$	177147

Table 2 shows the numerical results of the approximate calculation $I(f)$ using the cubature formula $\Phi cl(f)$, the approximation error $\varepsilon_2=\left|I(f)-\Phi cl(I)\right|$ and the number of used function values Q.

Table. 2. Calculation $I(f)$ according to the formula $\Phi cl(f)$

ℓ	$\sqrt{\ell^3}$	ℓ^3	ε_2	$Q=N=\ell^9$
4	8	64	$2.68\cdot 10^{-5}$	262144
9	27	729	$2.06\cdot 10^{-7}$	387420489

The numerical experiment was carried out in PTC MathCad 15.

CONCLUSION

Numerical integration of functions of many variables is widely used in mathematical modeling of systems and processes. It is time to develop cubature formulas for the approximate calculation of multiple integrals with different types of experimental data. Constructed cubature formulas use function values on planes and lines. Such cubature cubature formulas use new information operators.

Economic spline interpolation schemes of functions of three variables have been created on the basis of new information operators. Application of such schemes to the numerical integration of functions of three variables allows efficient calculation of triple integrals.

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