



CHARACTERIZATIONS OF K-HYPERGENERALIZED PROJECTORS

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Abstract

The paper focuses on the classes of the k-hypergeneralized projectors and, especially hypergeneralized projectors. Several features of these classes are identified, and properties are characterized.

Keywords: k-hypergeneralized projector, hypergeneralized projector.

INTRODUCTION

Let $\mathbb{C}^{n \times m}$ denote the set of all $n \times m$ complex matrices. For a matrix $A \in \mathbb{C}^{n \times m}$, the symbols A^* , R(A) and r(A) will stand for the conjugate transpose matrix, range and rank of A, respectively. Also, for a matrix $A \in \mathbb{C}^{n \times n}$, we denote by tr(A) and $\sigma(A)$ the trace and the spectrum of A, respectively. Henceforth, for $k \in \mathbb{N}$ and k > 1, the set of complex roots of 1 shall be denoted by σ_k and if we set $\omega_k = e^{2\pi i/k}$, then $\sigma_k =$ $\{\omega_k^0, \omega_k^1, \dots, \omega_k^{k/1}\}$. By I_n we will represent the identity matrix of order n. We denote that $A^0 = I_n$, for $A \in \mathbb{C}^{n \times n}$. The matrix $P \in \mathbb{C}^{n \times n}$ satisfying $P^2 = P$ is

The matrix $P \in \mathbb{C}^{n \times n}$ satisfying $P^2 = P$ is called the projector (the idempotent matrix), until the matrix $P \in \mathbb{C}^{n \times n}$ satisfying $P^2 =$ $P = P^*$ is called the orthogonal projector. A matrix $B \in \mathbb{C}^{n \times n}$ is said to be similar to a matrix $A \in \mathbb{C}^{n \times n}$ if there exists a nonsingular matrix $P \in \mathbb{C}^{n \times n}$ such that $B = P^{-1}AP$. If a matrix $A \in \mathbb{C}^{n \times n}$ is similar to a diagonal matrix, then A is said to be diagonalizable. The Moore-Penrose inverse of A is the unique matrix A^{\dagger} satisfying the equations:

(1) $AA^{\dagger}A = A$, (2) $A^{\dagger}AA^{\dagger} = A^{\dagger}$,

(3) $(AA^{\dagger})^* = AA^{\dagger}$, (4) $(A^{\dagger}A)^* = A^{\dagger}A$. The EP matrix (the range-Hermitian matrix) is the matrix $A \in \mathbb{C}^{n \times n}$ such that $A^{\dagger}A = A A^{\dagger}$, ie. $R(A) = R(A^*)$. The index of a matrix $A \in \mathbb{C}^{n \times n}$, is the smallest nonnegative integer k such that $r(A^{k+1}) = r(A^k)$, denoted by Ind(A). For $A \in \mathbb{C}^{n \times n}$, Ind(A) = k, the matrix $X \in \mathbb{C}^{n \times n}$ satisfying

(1^k)A^kXA = A^k, (2)XAX = X,(5) XA = AX

is called the Drazin inverse of A and is denoted by $X = A^d$. If Ind(A) = 1, then this special case of the Drazin inverse is known as the group inverse and is denoted by $A^{\#}$.

In 1997, Groβ and Trenkler [1] introduced hypergeneralized projectors:

the hypergenerelized projector is a square matrix such that $A^2 = A^{\dagger}$.

Later, in [2-6], different properties of hypergeneralized projector are given and finally by Tošić [7] who introduced khypergeneralized projectors defined by the following:

the k-hypergenerelized projector is a square matrix such that $A^k = A^{\dagger}$.

Different topics related to khypergeneralized projectors an, especially hypergenerlized projectors have been investigated extensively in the past decades. Inspired by the above-mentioned results, we will presents some characterizations of these classes of matrices are given in terms of the Moore-Penrose inverse, the EP matrix and



the conjugate transpose matrix, as well as appropriate matrix expressions.

K-HYPERGENERALIZED PROJECTORS

In this section, we give some characterizations of k-hypergeneralized projectors. First, we give necessary and sufficient conditions that A is a k-hypergeneralized projector.

Theorem 1. ([7]) Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

- (i) A is a k-hypergeneralized projector.
- (ii) A is an EP matrix, $\sigma(A) \subseteq \sigma_k(A) \cup \{0\}$ and A is diagonalizable.
- (iii) A is an EP matrix and $A^{k+2} = A$.
- (iv) A has the following representation $A = U \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} U^*$, where $U \in \mathbb{C}^{n \times n}$ is an unitary matrix and $K \in \mathbb{C}^{r \times r}$ is a nonsingular matrix such that $K^{k+1} = I_r$.

If A is a k-hypergeneralized projector, then A^{k+1} is the orthogonal projector onto R(A). Also, the converse implication is valid.

Theorem 2. ([7]) Let $A \in \mathbb{C}^{n \times n}$. Then A is a k-hypergeneralized projector if and only if A^{k+1} is the orthogonal projector onto R(A).

Corollary 3. ([7]) Let $A \in \mathbb{C}^{n \times n}$ be a k-hypergeneralized projector. Then $r(A) = tr(A^{k+1})$.

The next result represents necessary and sufficient conditions for $A \in \mathbb{C}^{n \times n}$ be a k-hypergeneralized projector.

Theorem 4. ([7]) Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

- (i) A is a k-hypergeneralized projector.
- (ii) A^* is a k-hypergeneralized projector.
- (iii) A^{\dagger} is a k-hypergeneralized projector.

Notice that if A is a k-hypergeneralized projector, then $A^{\dagger} = A^{\#} = A^{d} = A^{k} = A^{m(k+1)+k}, m \in \mathbb{N}.$

The following theorem singles out a sufficient condition for the equivalence of A being a k-hypergeneralized projector and A being an EP matrix.

Theorem 5. ([7]) Let $A \in \mathbb{C}^{n \times n}$. Assume there exists $B \in \mathbb{C}^{n \times n}$ such that B is a khypergeneralized projector and $A^2 = AB$ or $A^2 = BA$. Then A is a k-hypergeneralized projector if and only if A is an EP matrix.

In the following theorem, we give several characterizations of k-hypergeneralized projectors.

Theorem 6. ([8]) Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

- (i) *A* is a k-hypergeneralized projector.
- (ii) $A^{k+1} = A^{\dagger}A$.
- (iii) $A^{k+1} = AA^{\dagger}$.

(iv)
$$A^{k+1}A^{\dagger} = A^{\dagger}.$$

$$(\mathbf{v}) \qquad A^{\dagger}A^{k+1} = A^{\dagger}.$$

The next result implies that khypergeneralized projectors can be characterized by some equalities involving the conjugate transpose matrix.

Theorem 7. Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

(i) A is a k-hypergeneralized projector.

(ii)
$$A = A^{k+2}, (A^{k+1})^* = A^{k+1}.$$

(iii)
$$A = A^{k+2}, AA^{\dagger} = A^{\dagger}A.$$

(iv)
$$A = A^{k+2}, A^{\dagger}AAA^{\dagger} = AA^{\dagger}A^{\dagger}A.$$

(v) $A^* = A^{k+1}A^*$.

(vi)
$$A^* = A^* A^{k+1}$$
.

(vii)
$$A = (A^*)^{k+1}A$$
.

- (viii) $A = A(A^*)^{k+1}$.
- (ix) $A = A^{k+2}, A^{\dagger} = A(A^{*})^{k+1}.$
- (x) $A = A^{k+2}, A^{\dagger} = (A^*)^{k+1}A.$

Proof. If A is a k-hypergeneralized projector, then it commutes with A^{\dagger} and $A^{\dagger} = A^{k}$. It is not difficult to verify that conditions (ii)-(x) hold.

(ii) \Rightarrow (i) Suppose that $A = A^{k+2}$ and $(A^{k+1})^* = A^{k+1}$. Then we have

$$AA^{k}A = A^{k+2} = A,$$

$$A^{k}AA^{k} = A^{k+2}A^{k-1} = AA^{k-1} = A^{k},$$

$$(AA^{k})^{*} = (A^{k+1})^{*} = A^{k+1} = AA^{k},$$

$$(A^{k}A)^{*} = (A^{k+1})^{*} = A^{k+1} = A^{k}A.$$

Hence, $A^{\dagger} = A^k$, ie. A is a hypergeneralized projector.

(iii) \Rightarrow (i) From $A = A^{k+2}$ and $AA^+ = A^+A$, we obtained

$$A^{k} = AA^{k-2}A$$

= $(AA^{\dagger}A)A^{k-2}(AA^{\dagger}A)$
= $A^{\dagger}A^{k+2}A^{\dagger} = A^{\dagger}AA^{\dagger} = A^{\dagger}$

Thus, A is a k-hypergeneralized projector. (iv) \Rightarrow (iii) The equalities $A^{\dagger}AAA^{\dagger} = AA^{\dagger}A^{\dagger}A$ and $A^{k+2} = A$ give $A^{\dagger}A = A^{\dagger}A^{k+2} = A^{\dagger}AAA^{k} =$ $A^{\dagger}A(AA^{\dagger}A)A^{k} = (A^{\dagger}AAA^{\dagger})AA^{k} =$ $(AA^{\dagger}A^{\dagger}A)AA^{k} = AA^{\dagger}A^{\dagger}(AAA^{k}) =$ $AA^{\dagger}A^{\dagger}A^{k+2} = AA^{\dagger}A^{\dagger}A.$

Similarly, we obtained $AA^{\dagger} = AA^{\dagger}AA$. Since, $A = A^{k+2}$, we conclude that the condition (iii) holds.

(v) \Rightarrow (ii) Applying involution to the equality $A^* = A^{k+1}A^*$, we conclude that $A = A(A^{k+1})^*$. Also, by (v), we get

 $A^{k+1} = A^{k}A = A^{k}(A^{*})^{*} = A^{k}(A^{k+1}A^{*})^{*} = A^{k}A(A^{k+1})^{*} = A^{k+1}A^{*}(A^{k})^{*} = A^{*}(A^{k})^{*} = (A^{k+1})^{*}.$ Now, $A = A(A^{k+1})^{*} = AA^{k+1} = A^{k+2}.$

Therefore, the condition (ii) is satisfied.

(vi) \Rightarrow (ii) This part can be proved in a similar way as (v) \Rightarrow (ii).

(vii) \Rightarrow (vi) Applying involution to the equality $A = (A^*)^{k+1}A$, we get $A^* = A^*A^{k+1}$. Hence, the equality (vi) follows from the equality (vii).

 $(viii) \Rightarrow (v)$ This follows similarly as in the part $(vii) \Rightarrow (vi)$.

(ix) \Rightarrow (v) It is well-known that $A^* = A^{\dagger}AA^* = A^*AA^{\dagger}$. Now, if $A^{\dagger} = A(A^*)^{k+1}$ and $A = A^{k+2}$ hold, then

$$A^{k+2}A^* = A^{k+1}AA^* = A^{k+1}A^{\dagger}AA^*$$

= $A^{k+1}A(A^*)^{k+1}AA^*$
= $A^{k+2}(A^*)^{k+1}AA^*$
= $A(A^*)^{k+1}AA^* = A^{\dagger}AA^*$
= $A^*.$

Thus, the condition (v) are proved.

 $(x) \Rightarrow (vi)$ This implication can be obtained in the same manner as in $(ix) \Rightarrow (v)$.

HYPERGENERALIZED PROJECTORS

If we specialize to k=2 in the previous similarly, we obtained results for hypergeneralized projectors. The following results are given in [1] and [4].

Theorem 8. Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

- (i) *A* is a hypergeneralized projector.
- (ii) A is an EP matrix, $\sigma(A) \subseteq \sigma_2(A) \cup \{0\}$ and A is diagonalizable.
- (iii) A is an EP matrix and $A^4 = A$.
- (iv) A has the following representation $A = U \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} U^*$, where $U \in \mathbb{C}^{n \times n}$ is an unitary matrix and $K \in \mathbb{C}^{r \times r}$ is a nonsingular matrix such that $K^3 = I_r$.

Theorem 9. Let $A \in \mathbb{C}^{n \times n}$. Then A is a hypergeneralized projector if and only if A^3 is the orthogonal projector onto R(A).

Corollary 10. Let $A \in \mathbb{C}^{n \times n}$ be a hypergeneralized projector. Then $r(A) = tr(A^3)$.

Theorem 11. Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

- (i) *A* is a hypergeneralized projector.
- (ii) A^* is a hypergeneralized projector.
- (iii) A^{\dagger} is a hypergeneralized projector.

Notice that if A is a k-hypergeneralized projector, then $A^{\dagger} = A^{\#} = A^{d} = A^{2} = A^{3m+2}, m \in \mathbb{N}.$

Theorem 12. Let $A \in \mathbb{C}^{n \times n}$. Assume there exists $B \in \mathbb{C}^{n \times n}$ such that B is a hypergeneralized projector and $A^2 = AB$ or $A^2 = BA$. Then A is a hypergeneralized projector if and only if A is an EP matrix.

The following theorem is the corollary of Theorem 6 for k=2. It is given some necessary and sufficient conditions for characterizations of hypergeneralized projectors.

Theorem 13. Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

- (i) *A* is a hypergeneralized projector.
- (ii) $A^3 = A^{\dagger}A$.

(iii)
$$A^{3} = AA^{\dagger}$$
.
(iv) $A^{3}A^{\dagger} = A^{\dagger}$.
(v) $A^{\dagger}A^{3} = A^{\dagger}$.

The next result implies that hypergeneralized projectors can be characterized by some equalities involving the conjugate transpose matrix. It is the corollary of Theorem 7 for k=2.

Theorem 14. Let $A \in \mathbb{C}^{n \times n}$. Then the following statements are equivalent:

- (i) *A* is a hypergeneralized projector.
- (ii) $A = A^4, (A^3)^* = A^3.$
- (iii) $A = A^4, AA^\dagger = A^\dagger A.$
- (iv) $A = A^4$, $A^{\dagger}AAA^{\dagger} = AA^{\dagger}A^{\dagger}A$.
- $(\mathbf{v}) \qquad A^* = A^3 A^*.$
- (vi) $A^* = A^* A^3$.
- (vii) $A = (A^*)^3 A$.
- (viii) $A = A(A^*)^3$.
- (ix) $A = A^4, A^{\dagger} = A(A^*)^3$.
- (x) $A = A^4, A^{\dagger} = (A^*)^3 A.$

CONCLUSION

In this paper, we considered khypergeneralized projectors and, especially hypergeneralized projectors. Precisely, we characterized k-hypergeneralized projectors and hypergenerealized projectors in terms of equations involving their adjoints, the EP matrix and different characterizations of the Moore-Penrose inverse. We conclude that neither the rank nor the properties of operator matrices are necessary for the k-hypergeneralized characterization of projectors and hypergenerealized projectors. The characterization of k-hypergeneralized projectors and, especially hypergeneralized projectors is significant because these types of matrices are an indispensable part of modern mathematics and science.

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