International Scientific Conference
UNITECH`2024

CHARACTERIZATIONS OF K-HYPERGENERALIZED PROJECTORS

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Abstract

The paper focuses on the classes of the k-hypergeneralized projectors and, especially hypergeneralized projectors. Several features of these classes are identified, and properties are characterized.

Keywords: k-hypergeneralized projector, hypergeneralized projector.

INTRODUCTION

Let $\mathbb{C}^{n\times m}$ denote the set of all $n\times m$ complex matrices. For a matrix $A\in\mathbb{C}^{n\times m}$, the symbols A^*, $R(A)$ and $r(A)$ will stand for the conjugate transpose matrix, range and rank of A, respectively. Also, for a matrix $A\in\mathbb{C}^{n\times n}$, we denote by $tr(A)$ and $\sigma(A)$ the trace and the spectrum of A, respectively. Henceforth, for $k\in\mathbb{N}$ and $k>1$, the set of complex roots of 1 shall be denoted by σ_k and if we set $\omega_k=e^{2\pi i/k}$, then $\sigma_k=\{\omega_k^0,\omega_k^1,\dots,\omega_k^{k/1}\}$. By I_n we will represent the identity matrix of order n. We denote that $A^0=I_n$, for $A\in\mathbb{C}^{n\times n}$.

The matrix $P\in\mathbb{C}^{n\times n}$ satisfying $P^2=P$ is called the projector (the idempotent matrix), until the matrix $P\in\mathbb{C}^{n\times n}$ satisfying $P^2=P=P^*$ is called the orthogonal projector. A matrix $B\in\mathbb{C}^{n\times n}$ is said to be similar to a matrix $A\in\mathbb{C}^{n\times n}$ if there exists a nonsingular matrix $P\in\mathbb{C}^{n\times n}$ such that $B=P^{-1}AP$. If a matrix $A\in\mathbb{C}^{n\times n}$ is similar to a diagonal matrix, then A is said to be diagonalizable.

The Moore-Penrose inverse of A is the unique matrix $A^\dagger$ satisfying the equations:

$$(1)\ AA^\dagger A=A, (2)\ A^\dagger AA^\dagger=A^\dagger,$$
$$(3)\ (AA^\dagger)^*=AA^\dagger, (4)\ (A^\dagger A)^*=A^\dagger A.$$

The EP matrix (the range-Hermitian matrix) is the matrix $A\in\mathbb{C}^{n\times n}$ such that $A^\dagger A=A\,A^\dagger$, ie. $R(A)=R(A^*)$.

The index of a matrix $A\in\mathbb{C}^{n\times n}$, is the smallest nonnegative integer k such that $r(A^{k+1})=r(A^k)$, denoted by $Ind(A)$. For $A\in\mathbb{C}^{n\times n}$, $Ind(A)=k$, the matrix $X\in\mathbb{C}^{n\times n}$ satisfying

$$(1^k)A^kXA=A^k, (2)XAX=X,$$
$$(5)\ XA=AX$$

is called the Drazin inverse of A and is denoted by $X=A^d$. If $Ind(A)=1$, then this special case of the Drazin inverse is known as the group inverse and is denoted by $A^\#$.

In 1997, Groβ and Trenkler [1] introduced hypergeneralized projectors:

the hypergenerelized projector is a square matrix such that $A^2=A^\dagger$.

Later, in $[2-6]$, different properties of hypergeneralized projector are given and finally by Tošić [7] who introduced k-hypergeneralized projectors defined by the following:

the k-hypergenerelized projector is a square matrix such that $A^k=A^\dagger$.

Different topics related to k-hypergeneralized projectors an, especially hypergenerlized projectors have been investigated extensively in the past decades. Inspired by the above-mentioned results, we will presents some characterizations of these classes of matrices are given in terms of the Moore-Penrose inverse, the EP matrix and

doi: www.doi.org/10.70456/.........................

the conjugate transpose matrix, as well as appropriate matrix expressions.

K-HYPERGENERALIZED PROJECTORS

In this section, we give some characterizations of k-hypergeneralized projectors. First, we give necessary and sufficient conditions that A is a k-hypergeneralized projector.

Theorem 1. ([7]) Let $A \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:

(i) A is a k-hypergeneralized projector.

(ii) A is an EP matrix, $\sigma(A) \subseteq \sigma_k(A) \cup \{0\}$ and A is diagonalizable.

(iii) A is an EP matrix and $A^{k+2} = A$.

(iv) A has the following representation $A = U\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}U^*$, where $U \in \mathbb{C}^{n\times n}$ is an unitary matrix and $K \in \mathbb{C}^{r\times r}$ is a nonsingular matrix such that $K^{k+1} = I_r$.

If A is a k-hypergeneralized projector, then A^{k+1} is the orthogonal projector onto $R(A)$. Also, the converse implication is valid.

Theorem 2. ([7]) Let $A \in \mathbb{C}^{n\times n}$. Then A is a k-hypergeneralized projector if and only if A^{k+1} is the orthogonal projector onto $R(A)$.

Corollary 3. ([7]) Let $A \in \mathbb{C}^{n\times n}$ be a k-hypergeneralized projector. Then $r(A) = tr(A^{k+1})$.

The next result represents necessary and sufficient conditions for $A \in \mathbb{C}^{n\times n}$ be a k-hypergeneralized projector.

Theorem 4. ([7]) Let $A \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:

(i) A is a k-hypergeneralized projector.

(ii) A^* is a k-hypergeneralized projector.

(iii) $A^\dagger$ is a k-hypergeneralized projector.

Notice that if A is a k-hypergeneralized projector, then $A^\dagger = A^\# = A^d = A^k = A^{m(k+1)+k}, m \in \mathbb{N}$.

The following theorem singles out a sufficient condition for the equivalence of A being a k-hypergeneralized projector and A being an EP matrix.

Theorem 5. ([7]) Let $A \in \mathbb{C}^{n\times n}$. Assume there exists $B \in \mathbb{C}^{n\times n}$ such that B is a k-hypergeneralized projector and $A^2 = AB$ or $A^2 = BA$. Then A is a k-hypergeneralized projector if and only if A is an EP matrix.

In the following theorem, we give several characterizations of k-hypergeneralized projectors.

Theorem 6. ([8]) Let $A \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:

(i) A is a k-hypergeneralized projector.

(ii) $A^{k+1} = A^\dagger A$.

(iii) $A^{k+1} = AA^\dagger$.

(iv) $A^{k+1}A^\dagger = A^\dagger$.

(v) $A^\dagger A^{k+1} = A^\dagger$.

The next result implies that k-hypergeneralized projectors can be characterized by some equalities involving the conjugate transpose matrix.

Theorem 7. Let $A \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:

(i) A is a k-hypergeneralized projector.

(ii) $A = A^{k+2}, (A^{k+1})^* = A^{k+1}$.

(iii) $A = A^{k+2}, AA^\dagger = A^\dagger A$.

(iv) $A = A^{k+2}, A^\dagger AAA^\dagger = AA^\dagger A^\dagger A$.

(v) $A^* = A^{k+1}A^*$.

(vi) $A^* = A^*A^{k+1}$.

(vii) $A = (A^*)^{k+1}A$.

(viii) $A = A(A^*)^{k+1}$.

(ix) $A = A^{k+2}, A^\dagger = A(A^*)^{k+1}$.

(x) $A = A^{k+2}, A^\dagger = (A^*)^{k+1}A$.

Proof. If A is a k-hypergeneralized projector, then it commutes with $A^\dagger$ and $A^\dagger = A^k$. It is not difficult to verify that conditions (ii)-(x) hold.

(ii) $\Longrightarrow$ (i) Suppose that $A = A^{k+2}$ and $(A^{k+1})^* = A^{k+1}$. Then we have

$$AA^kA = A^{k+2} = A,$$
$$A^kAA^k = A^{k+2}A^{k-1} = AA^{k-1} = A^k,$$
$$(AA^k)^* = (A^{k+1})^* = A^{k+1} = AA^k,$$
$$(A^kA)^* = (A^{k+1})^* = A^{k+1} = A^kA.$$

Hence, $A^{\dagger} = A^k$, ie. A is a hypergeneralized projector.

(iii)⟹ (i) From $A = A^{k+2}$ and $AA^{+} = A^{+}A$, we obtained

$$\begin{aligned} A^k &= AA^{k-2}A \\ &= (AA^{\dagger}A)A^{k-2}(AA^{\dagger}A) \\ &= A^{\dagger}A^{k+2}A^{\dagger} = A^{\dagger}AA^{\dagger} = A^{\dagger}. \end{aligned}$$

Thus, A is a k-hypergeneralized projector.

(iv) ⟹ (iii) The equalities $A^{\dagger}AAA^{\dagger} = AA^{\dagger}A^{\dagger}A$ and $A^{k+2} = A$ give

$A^{\dagger}A = A^{\dagger}A^{k+2} = A^{\dagger}AAA^k = A^{\dagger}A(AA^{\dagger}A)A^k = (A^{\dagger}AAA^{\dagger})AA^k = (AA^{\dagger}A^{\dagger}A)AA^k = AA^{\dagger}A^{\dagger}(AAA^k) = AA^{\dagger}A^{\dagger}A^{k+2} = AA^{\dagger}A^{\dagger}A.$

Similarly, we obtained $AA^{\dagger} = AA^{\dagger}AA$.

Since, $A = A^{k+2}$, we conclude that the condition (iii) holds.

(v)⟹ (ii) Applying involution to the equality $A^* = A^{k+1}A^*$, we conclude that $A = A(A^{k+1})^*$. Also, by (v), we get

$A^{k+1} = A^kA = A^k(A^*)^* = A^k(A^{k+1}A^*)^* = A^kA(A^{k+1})^* = A^{k+1}A^*(A^k)^* = A^*(A^k)^* = (A^{k+1})^*$.

Now, $A = A(A^{k+1})^* = AA^{k+1} = A^{k+2}$.

Therefore, the condition (ii) is satisfied.

(vi)⟹ (ii) This part can be proved in a similar way as (v) ⇒ (ii).

(vii) ⟹ (vi) Applying involution to the equality $A = (A^*)^{k+1}A$, we get $A^* = A^*A^{k+1}$. Hence, the equality (vi) follows from the equality (vii).

(viii) ⟹ (v) This follows similary as in the part (vii) ⟹ (vi).

(ix) ⟹ (v) It is well-known that $A^* = A^{\dagger}AA^* = A^*AA^{\dagger}$. Now, if $A^{\dagger} = A(A^*)^{k+1}$ and $A = A^{k+2}$ hold, then

$$\begin{aligned} A^{k+2}A^* = A^{k+1}AA^* &= A^{k+1}A^{\dagger}AA^* \\ &= A^{k+1}A(A^*)^{k+1}AA^* \\ &= A^{k+2}(A^*)^{k+1}AA^* \\ &= A(A^*)^{k+1}AA^* = A^{\dagger}AA^* \\ &= A^*. \end{aligned}$$

Thus, the condition (v) are proved.

(x) ⟹ (vi) This implication can be obtained in the same manner as in (ix) ⟹ (v).

HYPERGENERALIZED PROJECTORS

If we specialize to k=2 in the previous similarly, we obtained results for hypergeneralized projectors. The following results are given in [1] and [4].

Theorem 8. Let $A \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:

(i) A is a hypergeneralized projector.

(ii) A is an EP matrix, $\sigma(A) \subseteq \sigma_2(A) \cup \{0\}$ and A is diagonalizable.

(iii) A is an EP matrix and $A^4 = A$.

(iv) A has the following representation $A = U\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}U^*$, where $U \in \mathbb{C}^{n\times n}$ is an unitary matrix and $K \in \mathbb{C}^{r\times r}$ is a nonsingular matrix such that $K^3 = I_r$.

Theorem 9. Let $A \in \mathbb{C}^{n\times n}$. Then A is a hypergeneralized projector if and only if A^3 is the orthogonal projector onto $R(A)$.

Corollary 10. Let $A \in \mathbb{C}^{n\times n}$ be a hypergeneralized projector. Then $r(A) = tr(A^3)$.

Theorem 11. Let $A \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:

(i) A is a hypergeneralized projector.

(ii) A^* is a hypergeneralized projector.

(iii) $A^{\dagger}$ is a hypergeneralized projector.

Notice that if A is a k-hypergeneralized projector, then $A^{\dagger} = A^{\#} = A^d = A^2 = A^{3m+2}, m \in \mathbb{N}$.

Theorem 12. Let $A \in \mathbb{C}^{n\times n}$. Assume there exists $B \in \mathbb{C}^{n\times n}$ such that B is a hypergeneralized projector and $A^2 = AB$ or $A^2 = BA$. Then A is a hypergeneralized projector if and only if A is an EP matrix.

The following theorem is the corollary of Theorem 6 for k=2. It is given some necessary and sufficient conditions for characterizations of hypergeneralized projectors.

Theorem 13. Let $A \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:

(i) A is a hypergeneralized projector.

(ii) $A^3 = A^{\dagger}A$.

(iii) $A^3 = AA^\dagger$.
(iv) $A^3 A^\dagger = A^\dagger$.
(v) $A^\dagger A^3 = A^\dagger$.

The next result implies that hypergeneralized projectors can be characterized by some equalities involving the conjugate transpose matrix. It is the corollary of Theorem 7 for k=2.

Theorem 14. Let $A \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:

(i) A is a hypergeneralized projector.
(ii) $A = A^4, (A^3)^* = A^3$.
(iii) $A = A^4, AA^\dagger = A^\dagger A$.
(iv) $A = A^4, A^\dagger AAA^\dagger = AA^\dagger A^\dagger A$.
(v) $A^* = A^3 A^*$.
(vi) $A^* = A^* A^3$.
(vii) $A = (A^*)^3 A$.
(viii) $A = A(A^*)^3$.
(ix) $A = A^4, A^\dagger = A(A^*)^3$.
(x) $A = A^4, A^\dagger = (A^*)^3 A$.

CONCLUSION

In this paper, we considered k-hypergeneralized projectors and, especially hypergeneralized projectors. Precisely, we characterized k-hypergeneralized projectors and hypergenerealized projectors in terms of equations involving their adjoints, the EP matrix and different characterizations of the Moore-Penrose inverse. We conclude that neither the rank nor the properties of operator matrices are necessary for the characterization of k-hypergeneralized projectors and hypergenerealized projectors. The characterization of k-hypergeneralized projectors and, especially hypergeneralized projectors is significant because these types of matrices are an indispensable part of modern mathematics and science.

Acknowledgments: We are grateful of the referee for helpful comments concerning the paper.
Authors are supported by Grant No. 451-03-65/2024-03/200123 of the Ministry of Science, Technological Development and Innovation, Republic of Serbia, and the project IJ-2303 of Faculty of Sciences and Mathematics, University of Priština in Kosovska Mitrovica.

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