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# A NONLINEAR APPROACH TO TEACHING THE MEAN VALUE THEOREMS PART II: NEW THEOREMS AND GRAPHICAL REPRESENTATIONS

**Valentina Todorova-Lazarova*, Penka Ivanova, Radostin Lefterov**

*Technical University of Gabrovo, Bulgaria*

** Corresponding author: lazarova@tugab.bg*

**Abstract**

*This paper presents a nonlinear approach to teaching the Mean Value Theorems (MVTs), which are recognized as crucial for mastering calculus. Students experience significant struggles in understanding the proofs and applications of MVTs due to traditional linear teaching methods. A nonlinear teaching methodology is proposed to address this issue, emphasizing interconnected understanding among MVTs and fostering conceptual clarity and critical thinking. Graphical representations are utilized, and new theorems are explored, leading to engagement in collaborative and interactive learning environments. New proofs are introduced, establishing relations among various MVTs, including Cauchy's Mean Value Theorem involving three functions. It is demonstrated that this approach significantly improves students' comprehension and ability to apply MVTs to complex problems. Practical implications suggest that diverse teaching strategies should be incorporated by educators to facilitate deeper learning. This research provides a novel framework for understanding MVTs, emphasizing their relevance and application in calculus.*

**Keywords:** mean value theorems, nonlinear teaching approach, graphical representations, new theorems, calculus education, theorem integration, mathematical understanding.

## I. INTRODUCTION

In recent years, the MVTs have been extensively discussed and explored. Many authors have extended the theorems to more complex contexts, such as higher dimensions or vector fields [1].

There has been a consistent focus on improving the teaching of MVTs in calculus courses [2].

Also, the MVTs have been applied in diverse fields, including multivariable calculus, physics, engineering, and finance [3, 4, 5, 6].

In the previous paper, a series of relations between MVTs was established, and new proofs that strengthen existing results were introduced [7]. These new proofs offer further insights into the fundamental principles of the theorems, contributing to a better understanding of their implications.

Nonlinear teaching methods aim to give students a better understanding of mathematical concepts. This approach fosters critical thinking, and creativity rather than relying on rote memorization or traditional methods. The basic aims of our research include:

- conceptual understanding, which involves guiding students to comprehend the geometric meaning of the theorems;
- exploration of generalized and new theorems, encouraging students to explore variations of classical theorems, generalize MVTs to new forms and make proofs of new theorems;
- collaborative and interactive learning, exploring theorems through software, students can visualize various functions and graphing them to find where the theorems hold;
- fostering theoretical discovery, encouraging students to explore if MVTs hold for complex functions;
- inspiring creativity, which leads to deep understanding, some personal discoveries, and new perspectives on established results.




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## II. NONLINEAR TEACHING APPROACH AND GRAPHICAL REPRESENTATIONS

Nonlinear teaching approach of MVTs provides many benefits by engaging students through diverse methods, and interpretations, which enhance both comprehension and application. This approach aligns with teaching strategies that encourage active learning, critical thinking, and a conceptual understanding of mathematics. It helps students to apply MVTs more effectively by emphasizing interconnections between these theorems (fig. 1).

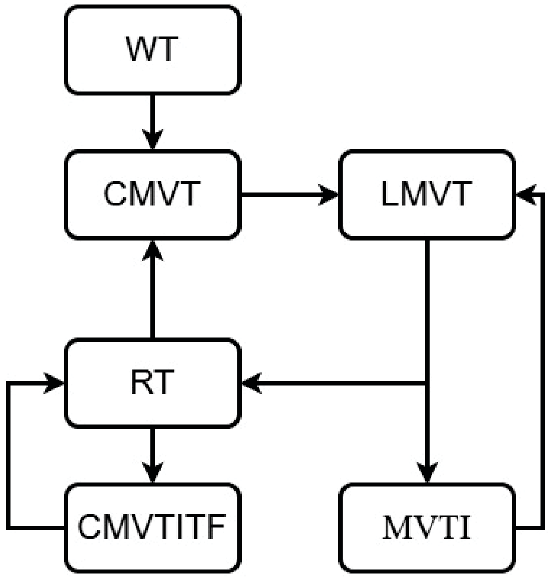


***Fig. 1*** *Relations between MVTs*

Understanding the connection between Roll's theorem (RT) [7, Theorem 2] and Cauchy's mean value theorem involving three functions (CMVTITF) [7, Theorem 6] includes recognizing how CMVTITF generalizes the concepts found in RT.

**Theorem 1**. Cauchy's mean value theorem involving three functions follows from Roll's theorem.

Proof. We consider an auxiliary function, defined on the closed interval $[a, b]$

$$\Delta(x) = \begin{Vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{Vmatrix}.$$

This function satisfies all statements in RT:

- $\Delta(x)$ is continuous on the closed interval $[a, b]$, as if $f(x)$, $g(x)$ and $h(x)$ are continuous;
- $\Delta(x)$ is differentiable on the open interval $(a, b)$, as if $f(x)$, $g(x)$ and $h(x)$ are differentiable;
- $\Delta(a) = \Delta(b) = 0$.

We differentiate $\Delta(x)$ with respect to $x$

$$\Delta'(x) = \begin{Vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{Vmatrix} + \begin{Vmatrix} f(x) & g(x) & h(x) \\ f'(a) & g'(a) & h'(a) \\ f(b) & g(b) & h(b) \end{Vmatrix} + \begin{Vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f'(b) & g'(b) & h'(b) \end{Vmatrix}$$
$$= \big(g(a)h(b) - h(a)g(b)\big)f'(x) + \big(h(a)f(b) - f(a)h(b)\big)g'(x) + \big(f(a)g(b) - g(a)f(b)\big)h'(x).$$

According to RT, there exists a point $\xi$ in $(a, b)$ such that

$$\Delta'(\xi) = \big(g(a)h(b) - h(a)g(b)\big)f'(\xi) + \big(h(a)f(b) - f(a)h(b)\big)g'(\xi) + \big(f(a)g(b) - g(a)f(b)\big)h'(\xi) = 0 \quad (1)$$

We reorganize (1)

$$\Delta'(\xi) = \frac{\big(g(a)h(b) - h(a)g(b)\big)\big(f(b) - f(a)\big)}{f(b) - f(a)} f'(\xi) + \frac{\big(h(a)f(b) - f(a)h(b)\big)\big(g(b) - g(a)\big)}{g(b) - g(a)} g'(\xi) + \frac{\big(f(a)g(b) - g(a)f(b)\big)\big(h(b) - h(a)\big)}{h(b) - h(a)} h'(\xi) = 0$$

Since $f(a), f(b), g(a), g(b), h(a)$ and $h(b)$ are constants, we can make the subsequent substitutions:

$$k_1 = \big(g(a)h(b) - h(a)g(b)\big)\big(f(b) - f(a)\big);$$
$$k_2 = \big(h(a)f(b) - f(a)h(b)\big)\big(g(b) - g(a)\big);$$
$$k_3 = \big(f(a)g(b) - g(a)f(b)\big)\big(h(b) - h(a)\big),$$

in which $k_1$, $k_2$ and $k_3$ are three real numbers.

Then it is checked $k_1 + k_2 + k_3 = 0$. Thus

$$\frac{k_1}{f(b) - f(a)} f'(\xi) + \frac{k_2}{g(b) - g(a)} g'(\xi) + \frac{k_3}{h(b) - h(a)} h'(\xi) = 0. \blacksquare \quad (2)$$

To show how CMVTITF connects to RT, we need to prove that RT is a special case of this general result.

**Theorem 2**. Roll's theorem follows from Cauchy's mean value theorem involving three functions.

Proof. Consider $f(x)$ is a function satisfying all conditions stated in CMVTITF. Let us define two functions $g(x)$ and $h(x)$ such that $g(x) = h(x) = x$. Obviously $g(x)$ and $h(x)$ are continuous on $[a, b]$ and differentiable on $(a, b)$, and $g'(x) = h'(x) = 1$ for all $x$. We substitute $g(b) = h(b) = b$ and $g(a) = h(a) = a$ into (2). We obtain

$$\frac{k_1}{f(b) - f(a)} f'(\xi) + \frac{k_2}{b - a} + \frac{k_3}{b - a} = 0 \quad (3)$$

We denominate (3)

$$k_1(b - a)f'(\xi) + k_2\big(f(b) - f(a)\big) + k_3\big(f(b) - f(a)\big) = 0$$

Since the values of the function $f(x)$ at the endpoints of $[a, b]$ are equal, i.e. $f(a) = f(b)$ in RT, we have $k_1(b - a)f'(\xi) = 0$ Thus $f'(\xi) = 0$. $\blacksquare$

By setting two of the three functions equal and applying the conditions of RT, we simplify CMVTITF to show that there exists a point where the derivative is equal to zero, so RT can be viewed as a special case of CMVTITF.

If any of the conditions, stated in MVTs are violated, some functions disprove the theorems. Here are some examples.

If a function is not continuous on a closed interval $[a, b]$, RT may not hold. Consider the function $f(x) = \frac{1}{x^2}$ in $[-1, 1]$. We have $f(-1) = f(1) = 1$. However, $f(x)$ is not continuous at $x = 0$, and there is no point $\xi$ in $(-1, 1)$ where $f'(\xi) = 0$ (fig. 2).

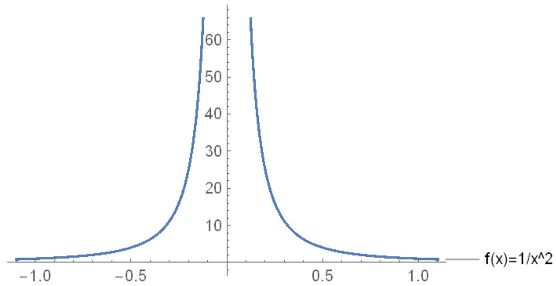


***Fig. 2.*** *Graph of the function* $f(x) = \frac{1}{x^2}$

If a function $f(x)$ is such that its derivative does not exist at all inner points of $[a, b]$, then the statement in (RT) may turn out invalid. For example, the function $f(x) = 1 - \sqrt[3]{x^2}$ is continuous in $[-1, 1]$, and it is equal to zero at the endpoints of the interval, i.e. $f(-1) = f(1) = 0$. However it is not differentiable at $x = 0$. There is no point $\xi$ in $(-1, 1)$ where $f'(\xi) = 0$, because the derivative $f'(x) = -\frac{2}{3\sqrt[3]{x}}$ does not exist at $x = 0$. Actually, $f'(0) = \infty$ (fig. 3).

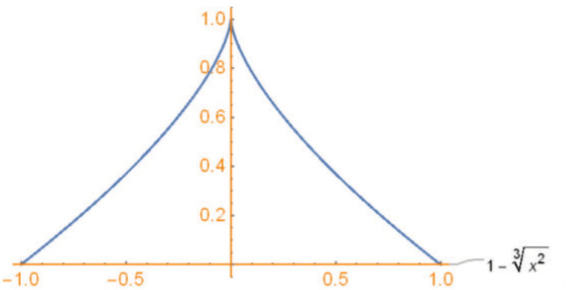


***Fig. 3.*** *Graph of the function* $f(x) = 1 - \sqrt[3]{x^2}$

## III. NEW THEOREMS

As it is known Cauchy's mean value theorem (CMVT) [7, Theorem 3] stated that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \qquad (4)$$

Here we present a new Cauchy's mean value theorem for three functions.

**Theorem 3**. Let the functions $f(x)$, $g(x)$ and $\varphi(x)$ be such that the functions $F(x) = \frac{f(x)}{\varphi(x)}$ and $G(x) = \frac{g(x)}{\varphi(x)}$ satisfy all conditions of CMVT in the closed interval $[a, b]$, then

$$\frac{f(b).\varphi(a) - f(a).\varphi(b)}{g(b).\varphi(a) - g(a).\varphi(b)} = \frac{f(\xi).\varphi'(\xi) - f'(\xi).\varphi(\xi)}{g(\xi).\varphi'(\xi) - g'(\xi).\varphi(\xi)}. \qquad (5)$$

Proof. Applying statement (4) we have

$$\frac{F'(\xi)}{G'(\xi)} = \frac{F(b) - F(a)}{G(b) - G(a)}$$

which is equivalent to

$$\frac{\left[\frac{f(x)}{\varphi(x)}\right]' / x = \xi}{\left[\frac{g(x)}{\varphi(x)}\right]' / x = \xi} = \frac{\frac{f(b)}{\varphi(b)} - \frac{f(a)}{\varphi(a)}}{\frac{g(b)}{\varphi(b)} - \frac{g(a)}{\varphi(a)}}. \qquad (6)$$

By simplifying both terms in (6), we obtain

$$\left[\frac{\frac{f'(x).\varphi(x) - f(x).\varphi'(x)}{\varphi^2(x)}}{\frac{g'(x).\varphi(x) - g(x).\varphi'(x)}{\varphi^2(x)}}\right]' \Big/ x = \xi = \frac{f(b).\varphi(a) - f(a).\varphi(b)}{g(b).\varphi(a) - g(a).\varphi(b)}.$$

By reducing the fraction in the latter equality we obtain (5). ■

The next theorem is a consequence from Theorem 3.

**Theorem 4.** Let the functions $f(x)$ and $g(x)$ be defined in a closed interval $[a, b]$ and functions $xf(x)$ and $xg(x)$ fulfil all conditions of CMVT.

Then the following identity is true

$$\frac{f(\xi) + \xi.f'(\xi)}{g(\xi) + \xi.g'(\xi)} = \frac{b.f(b) - a.f(a)}{b.g(b) - a.g(a)}.$$

Proof. We apply Theorem 3 with the function $\varphi(x) = \frac{1}{x}$.

So we have

$$\frac{f(\xi).\left(\frac{1}{\xi}\right)' - f'(\xi).\frac{1}{\xi}}{g(\xi).\left(\frac{1}{\xi}\right)' - g'(\xi).\frac{1}{\xi}} = \frac{f(b).\frac{1}{a} - f(a).\frac{1}{b}}{g(b).\frac{1}{a} - g(a).\frac{1}{b}}$$

We transform the upper equality in

$$\frac{f(\xi).\left(-\frac{1}{\xi^2}\right) - f'(\xi).\frac{1}{\xi}}{g(\xi).\left(-\frac{1}{\xi^2}\right) - g'(\xi).\frac{1}{\xi}} = \frac{bf(b) - af(a)}{bg(b) - ag(a)}.$$

By simplifying the left hand side of the last equality, we continue with

$$\frac{f(\xi).\frac{1}{\xi} + f'(\xi)}{g(\xi).\frac{1}{\xi} + g'(\xi)} = \frac{bf(b) - af(a)}{bg(b) - ag(a)}. \qquad (7)$$

Finally, by reducing (7), we obtain the validity of the theorem. ■

The next theorem, that is presented, follows from Lagrange's mean value theorem (LMVT) [7, Theorem 4].

**Theorem 5**. Let the function $f(x)$ satisfies the conditions of LMVT in the closed interval $[a, b]$ and $0 \notin [a, b]$.

Then the following identity is true

$$\frac{bf(a) - af(b)}{b - a} = f(\xi) - \xi f'(\xi).$$

Proof 1. Since the function $f(x)$ satisfies LMVT and $0 \notin [a, b]$, the functions

$$F(x) = \frac{f(x)}{x} \text{ and } G(x) = \frac{1}{x} \qquad (8)$$

satisfy the conditions of Theorem 3.
Note that in this case the function $g(x) = 1$, and $\varphi(x) = x$.
Then we have

$$\frac{f(b).a - f(a).b}{a - b} = \frac{f(\xi) - \xi f'(\xi)}{1.1 - 0.\xi}. \qquad (9)$$

It remains to multiply (9) by $(-1)$ in order to complete the proof. ■
Proof 2. From the conditions of the theorem follows that the functions $F(x)$ and function $G(x)$ defined in (8) satisfy the statement (4) in CMVT.
Therefore

$$\frac{\left[\frac{f(x)}{x}\right]' / x = \xi}{\left(\frac{1}{x}\right)' / x = \xi} = \frac{\frac{f(b)}{b} - \frac{f(a)}{a}}{\frac{1}{b} - \frac{1}{a}}. \qquad (10)$$

By reducing (10), we continue with

$$\frac{f'(\xi).\xi - f(\xi)}{-\frac{1}{\xi^2}} = \frac{af(b) - bf(a)}{a - b}.$$

And, finally by simplifying the left hand side of the upper equality, we have

$$f(\xi) - \xi f'(\xi) = \frac{af(b) - bf(a)}{a - b} = \frac{bf(a) - af(b)}{b - a}. ■$$

## IV. CONCLUSION

A nonlinear approach to teaching MVTs is an effective strategy in fostering a better conceptual understanding and engagement among students. By introducing graphical representations and the exploration of new theorems we have created a learning environment that encourages students to move beyond rote memorization to a more comprehensive and flexible grasp of these essential concepts.

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