THE SECOND MEAN VALUE THEOREM FOR COMPLEX LINE INTEGRAL

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Abstract

In real iterations, several types of mean value theorems for definite integrals are used. In complex domain, we cannot specifically formulate the mean value theorem of a particular complex line integral $\int_L f(z)\,dz$*, since we are unable to give an appropriate geometric interpretation of the integral over the surface below a curve* L *(from* z_0 *to* z_1*). Based on the mean value theorems for a complex line integral in [Vujakovic J., The mean value theorem of line complex integral and Sturm function. Applied Mathematical Sciences 2014; 8 (37): 1817-1827.], we got the idea to formulate the second mean value theorem in complex domain for the product of two analytic functions.*

Keywords: mean value theorem, analytic function, power series, iteration.

INTRODUCTION AND PRELIMINARIES

In real iteration several types of mean value theorems for definite integrals are used. We will mention only two.

Theorem 1. (The first mean value theorem [1]). Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then there exists $\xi\in(a,b)$ such that

$$\int_a^b f(x)\,dx = f(\xi)(b-a). \tag{1}$$

Theorem 2. (The second mean value theorem [2]) If $f:[a,b]\to\mathbb{R}$ is a continuous function on $[a,b]$ with $x\in(a,b)$, $m\le f(x)\le M$ and $g:[a,b]\to\mathbb{R}$ is an integrable function, then there exists ξ in (a,b) such that

$$\int_a^b f(x)\,g(x)\,dx = f(\xi)\int_a^b g(x)\,dx. \tag{2}$$

It is usual to use the term of mean values for complex integral and closed contours, according to which $f(z)$ is analytical or continuous while inside contour it can be discontinuous and even non-analytical [3] This was suggested by Cauchy's fundamental theorem

$$\oint_L f(z)\,dz = 0. \tag{3}$$

If z is interior point of contour L then Cauchy's integral formula is valid

$$f(z) = \frac{1}{2\pi i}\oint_L \frac{f(\xi)}{\xi - z}\,d\xi. \tag{4}$$

If $f(z)$ is analytical on L, then L can be replaced by the simplest closed contour, circle $K=K(z,R)$, so standard formula (4) is also valid for $L\equiv K$.

The mean value of function was obtained from (4), for $\xi = z+Re^{i\varphi}, 0\le\varphi\le 2\pi$, according to the following formula

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+Re^{i\varphi})\,d\varphi. \tag{5}$$

However, in solving complex differential equations and for the needs of iterations, we could not use the formulas from (3) to (5). It was necessary to find a simple formula of mean value, as in a real case [4], where ξ is a certain kind of mean value between points z_0 and z, and $a(z)$ is an analytic function. In [5] we have shown that formula

$$\int_{L:z_0}^{z} a(z)dz = a(\xi)(z-z_0) \qquad (6)$$

where $a(z)$ is analytical function in area G, where is placed curve L on which integration is conducted, z and z_0 are initial and final limits of integral, ξ is certain point from G, does not have to be L but which satisfies certain standards of approximation, can be adopted as some kind of mean value for complex integral if integration path is open line L.

In the further work we need the following statements.

Theorem 3 ([5]). The integral of power function $a(z)=z^n, n\geq 0$, on Jordan curve L, inside the circle $|z|\leq R$ can be replaced by integral on the shortest path, that is, on direction Oz,

$$\int_{L:0}^{z} a(z)dz = a(\xi)z \qquad (7)$$

where

$$\xi = \frac{z}{\sqrt[n]{n+1}} \qquad (8)$$

is the midpoint on the path and

$$a(\xi) = a(\frac{z}{\sqrt[n]{n+1}}) = \frac{z^n}{n+1}. \qquad (9)$$

Formula (7) is the mean value formula for complex integral.

Theorem 4. [5] For arbitrary polynomial $P_n(z)=\sum_{k=0}^{n} a_k z^k$ applies formula (7), for mean value of integral, where ξ is certain mean value of z with module lesser than $|z|=R$ and which depends on z, that is on $P_n(z)$.

MAIN RESULTS

For the needs of iterations, for solving a complex differential equation, we found important to find the complex line integral on the open path

$$I = \int_{L:0}^{z} za(z)dz \qquad (10)$$

where $f(z)=a(z)$ and $g(z)=z$.

Based on the theorems 1 and 2, together with the theorems 3 and 4, we got the idea to formulate the second mean value theorem for complex line integrals of product of analytical functions $f(z)$ and $g(z)$.

Theorem 5. For integrals of the type (10), where $a(z)$ is the analytical function in the circle $|z|\leq R$, the formula of the mean integral value is valid

$$\int_{L:0}^{z} za(z)dz = z^2 a(\xi) \qquad (11)$$

where ξ is the interior point of the circle $|\xi|<|z|=R$.

Proof. For simplicity, let analytical function $f(z)=a(z)$ be given by power series

$$a(z) = \sum_{k=0}^{+\infty} a_k z^k = a_0 + a_1 z + \cdots + a_k z^k + \cdots \ . \ (12)$$

Since

$$za(z) = \sum_{k=0}^{+\infty} a_k z^{k+1}$$
$$= a_0 z + a_1 z^2 + a_2 z^3 + \cdots + a_k z^{k+1} + \cdots,$$

for the sake of analyticity, this series can be integrated term by term. Hence, we obtain series $z^2 \sum_{k=0}^{+\infty} \frac{a_k}{k+2} z^k$.

On the other hand, using the mean value Theorem 3, that is formula (7), we obtain equality

$$z^2 \sum_{k=0}^{+\infty} \frac{a_k}{k+2} z^k = z^2 \sum_{k=0}^{+\infty} a_k \xi^k \ . \qquad (13)$$

Now, we try to find the relationship between the mean value ξ and the argument z, for each coefficient a_k. From equality (13)

$$z^2 \left[\frac{a_0}{2} + \frac{a_1 z}{3} + \frac{a_2 z^2}{4} + \cdots + \frac{a_k z^k}{k+2} + \cdots \right] =$$
$$= z^2 \left[a_0 + a_1 \xi + a_2 \xi^2 + \cdots + a_k \xi^k + \cdots \right]$$

follows

$$\frac{a_0}{2} = a_0, \frac{a_1 z}{3} = a_1 \xi, \frac{a_2 z^2}{4} = a_2 \xi^2, \ldots, \frac{a_k z^k}{k+2} = a_k \xi^k,$$

that is $a_0 = 0, \xi = \frac{z}{3} = \frac{z}{2} = \cdots = \frac{z}{\sqrt[k]{k+2}}$.

Since this is contradictory, we must observe the sums of analytic series $\sum_{k=0}^{+\infty}\frac{a_k}{k+2}z^k$ and $\sum_{k=0}^{+\infty}a_k\xi^k$. If we evaluate these series, we see that the coefficients of the modular series $\sum_{k=0}^{+\infty}\frac{|a_k|}{k+2}|z^k|$ are smaller than the coefficient of series $\sum_{k=0}^{+\infty}|a_k||\xi^k|$. The same can be expected for the sum, because the coefficients a_k directly affect the radius of convergence.

In order to maintain the equality of the modules of nth partial sums, as in the proof of Theorem 4, we conclude that between ξ and z there is a connection $|z|>|\xi|$, which was to be proven. ■

Theorem 6. The mean value for integral $\int_{L:0}^{z} z^k a(z)\,dz, k\in\square$, where $a(z)$ is the analytical function in the circle $|z|\le R$, ($|\xi|\le R$) is given by

$$\int_{L:0}^{z} z^k a(z)\,dz = z^{k+1}a(\xi). \tag{14}$$

Proof. From the development of analytical function $a(z)$ in power series (12) we have

$$\int_{L:0}^{z} z^k a(z)\,dz =$$

$$= z^{k+1}\left[\frac{a_0}{k+1}+\frac{a_1 z}{k+2}+\frac{a_2 z^2}{k+3}+\cdots+\frac{a_n z^n}{k+n+1}+\cdots\right]$$

$$\overset{(7)}{=} z^{k+1}\sum_{n=0}^{+\infty}a_n\xi^n.$$

Dividing by $z^{k+1}\ne 0$ and equating multipliers with a_k, we have $a_0=0$, and

$$\xi=\frac{z}{k+2}=\frac{z}{\sqrt{k+3}}=\ldots=\frac{z}{\sqrt[n]{k+n+1}}.$$

Since this is contradictory, by repeating the procedure as in the proof of Theorem 5, we conclude that formula (14) is valid for $|\xi|<|z|=R$. ■

Note that the last property also applies to multimorphic analytical functions $\sqrt{z},\sqrt[n]{z}$, $\ln z$, Artanh$z$ ... which are to a certain extent unambiguous because the mean values are inside the circle $|\xi|<|z|=R$.

For iterations it is also important to find a formula for the integral $\int_{L:0}^{z} f(z)g(z)\,dz$, of the product of two analytical functions. Assume that $a(z)=f(z)g(z)$. Then, according to formula (7), we have

$$\int_{L:0}^{z} f(z)g(z)\,dz = f(\xi)g(\xi)z, \tag{15}$$

for $|\xi|<|z|=R$.

However, if part of the integral in formula (15) is solved, for example integral $\int_L g(z)\,dz$, then we need some kind of Second mean value theorem for a complex integral.

Theorem 7 (The second mean value theorem for a complex integral) For an integral of the product of two analytical functions $f(z)$ and $g(z)$, along the open path L which connects points $z_0=0$ and z and which is contained in the circle of radius $|z|\le R$ valid formula

$$\begin{aligned}\int_{L:0}^{z} f(z)g(z)\,dz &= f(\xi)\int_{L:0}^{z_1} g(z)\,dz =\\ &= g(\xi)\int_{L:0}^{z_2} f(z)\,dz,\end{aligned} \tag{16}$$

where the modules $|\xi|,|z_1|,|z_2|$ are inside the circle $|z|\le R$.

Proof. If in (15) we perform a grouping in the following way

$$\int_{L:0}^{z} f(z)g(z)\,dz = f(\xi)\big(g(\xi)\cdot z\big), |\xi|<|z|$$

in brackets, we again have the case from Theorem 3, but with some ξ which is different from z. Without loss of generality, we can assume that $f(z)=1$. Then

$$g(\xi)z = \int_{L:0}^{z_1} g(z)dz, |\xi| < |z_1|,$$

so

$$\int_{L:0}^{z} f(z)g(z)dz = f(\xi)\int_{L:0}^{z_1} g(z)dz$$

where $|\xi|, |z_1| < |z| = R$. Analogously, if in (15) we make a grouping

$$\int_{L:0}^{z} f(z)g(z)dz = g(\xi)(f(\xi)z), |\xi| < |z|$$

we again have in brackets structure of the mean value for integral, but now to some point z_2 for which also applies

$$\int_{L:0}^{z} f(z)g(z)dz = g(\varsigma)\int_{L:0}^{z_2} f(z)dz,$$

where $|z_2| < |z| = R$. ■

Note that the same is true for multiple products

$$\int_{0}^{z} f(z)g(z)h(z)dz = f(\xi)g(\xi)h(\xi)z =$$

$$= f(\xi)\int_{0}^{z_1} g(z)h(z)dz =$$

$$= f(\xi)g(\mu)\int_{0}^{z_2} h(z)dz =$$

$$= f(\xi)g(\mu)h(\eta)z_2 = ...$$

where all intermediates $\xi, \mu, \eta, z_1, z_2 \ldots$ per module are less than $|z| \le R$.

CONCLUSION

When we solve the integrals used in iterations, using the above theorems, we can calculate successive double integrals.

- For a constant $a(\xi)$ we have

$$\int_{L:0}^{z}\int_{L:0}^{z} a(z)dz^2 = \int_{L:0}^{z}\left(\int_{L:0}^{z} a(z)dz\right)dz =$$

$$= \int_{L:0}^{z} a(\xi)zdz = a(\xi)\frac{z^2}{2},$$

$|\xi| < |z| \le R$.

- Based on the previous case, we have

$$\int_{L:0}^{z}\int_{L:0}^{z} a(z)dz^2 \int_{L:0}^{z}\int_{L:0}^{z} a(z)dz^2 =$$

$$= \int_{L:0}^{z}\int_{L:0}^{z} a(z)a(\xi)\frac{z^2}{2!}dz^2.$$

If we do not change the upper limit z of integral, then ξ is constant, $a(\xi)$ is also constant and according to the second mean value theorem, for $|\xi|, |\mu| < |z| \le R$ we get

$$\int_{L:0}^{z}\int_{L:0}^{z} a(z)a(\xi)\frac{z^2}{2!}dz^2 =$$

$$= a(\xi)\int_{L:0}^{z}\int_{L:0}^{z} a(z)\frac{z^2}{2!}dz^2$$

$$= a(\xi)a(\mu)\int_{L:0}^{z}\int_{L:0}^{z} \frac{z^2}{2!}dz^2$$

$$= a(\xi)a(\mu)\frac{z^4}{4!}.$$

In this way, the double integrals in the iterations are treated as in the real region, only that the arguments of the mean values are not on the segment $[0, x]$, but are in some circle $|z| \le R$.

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