COMPLEX HOMOGENEOUS DIFFERENTIAL EQUATION OF FIRST AND SECOND ORDER THROUGH ITERATIONS

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Abstract
The theory of complex differential equations represents an important mathematical discipline both from the theoretical point of view and from numerous applications. Its development was equally encouraged by mathematicians, physicists and engineers. By using the series-iterations method we have explained how the obtained results could be applied to the complex linear homogeneous first order and canonical complex linear second order differential equations. We have also formulated some conclusions about the determination of zero solutions of these equations. Even if that equation had no solutions we have found a particular integral which had zero for some choices of the integration constants.

Keywords: a series-iteration method, differential equation, zero of solutions.

INTRODUCTION AND PRELIMINARIES

The ordinary differential equations have been applied in many fields such as physics, engineering, technology, biology, chemistry and other sciences. This is a very powerful mathematical tool for analyzing the relationship between various phenomena in nature. Techniques to solve differential equations or systems of differential equations are not always straightforward.

The iterations are mainly used to solve differential equations with completely determined, relatively elementary and continuous coefficients. We noticed that there is no theorem on the existence of a solution which requires that the coefficients of the equation must be specifically determined. They only require continuity and Lipschitz's condition. This created a common practice that the coefficients in the differential equation must be very concrete, and with well-established operations therein. In fact, this is just a psychological deception, because iterations can be performed well even when the equation coefficients are general. Then, for the linear differential equations, nice symmetric iterations are obtained, which are transformed into series. This classic method of iterations is called a series-iteration method.

Differential equation of first order. The complex canonical first order differential equation, with an analytical coefficient $a(z)$

$$\frac{dw}{dz} = a(z)w(z) \tag{1}$$

is equivalent to an integral equation

$$w(z) = \int_{z_0}^{z} a(z)w(z)dz + w_0 \ , \tag{2}$$

which means that $w(z)$ is the solution of the starting equation if and only if it is the solution of the integral equation (2).

Based on the integral equation (2) we define a iteration sequence $\left\{w^{[n]}(z)\right\}$:

$$w^{[n]}(z) = w_0 + \int_{z_0}^{z} a(z)w^{[n-1]}(z)dz, n = 1,2,3,... \tag{3}$$

with initial conditions $w^{[0]}(z_0) = w(z_0) = w_0$. We choose the initial approximation arbitrarily, but so that $w^{[0]}(z)$ is a continuous function in the domain of iterations and that it approximately satisfies the equation. For members of the sequence we have

$$w^{[1]}(z)=\int_{z_0}^{z} a(z)w^{[0]}(z)dz+w_0=w_0\left[1+\int_{z_0}^{z} a(z)dz\right],$$

$$w^{[2]}(z)=\int_{z_0}^{z} a(z)w^{[1]}(z)dz+w_0=$$

$$=w_0\left[1+\int_{z_0}^{z} a(z)\,dz+\int_{z_0}^{z} a(z)\int_{z_0}^{z} a(z)\,dz^2\right],$$

$$\vdots$$

$$w^{[n]}(z)=w_0\sum_{k=0}^{n}\underbrace{\int_{z_0}^{z} a(z)\int_{z_0}^{z} a(z)\dots\int_{z_0}^{z} a(z)dz^k}_{k\text{-integrals}}\ . \qquad (4)$$

and so on.

This mathematical construction makes sense if a series of iterations $\left\{w^{[n]}(z)\right\}$ converges. It is easy to check that integral operator (3) is a contraction. We will use a modulus apparatus, because it is only possible to make estimates in the set of complex numbers.

First of all, assume that in the domain G, in the z-plane, there is a finite curve L on which we integrate. Since $a(z)$ is an analytic function in G, its modulus is bounded, that is, there exists a real $M>0$ such that $|a(z)|\le M$ for all $z\in G$. To facilitate calculations, we translate the coordinate system by assuming that $z_0=0$. Since

$$\left|w^{[1]}(z)-w^{[0]}(z)\right|=\left|w_0\int_0^z a(z)dz\right|$$

$$\le|w_0|\left(\max_{z\in G}|a(z)|\right)|z|\le M|w_0||z|$$

by mathematical induction it is easy to prove that

$$\left|w^{[n]}(z)-w^{[n-1]}(z)\right|=\left|w_0\underbrace{\int_0^z a(z)\int_0^z a(z)\dots\int_0^z a(z)dz^n}_{n\text{-integrals}}\right|$$

$$=|w_0|\frac{\left|\int_0^z a(z)dz\right|^n}{n!}\le|w_0|\left(\max_{z\in G}|a(z)|\right)^n\frac{|z|^n}{n!}$$

$$\le|w_0|\frac{\left(M|z|\right)^n}{n!}.$$

Now we construct a functional series

$$w^{[0]}(z)+\sum_{n=1}^{\infty}\left[w^{[n]}(z)-w^{[n-1]}(z)\right] \qquad (5)$$

which is majored by series $\sum_{n=0}^{\infty}|w_0|\frac{\left(M|z|\right)^n}{n!}$. Based on d'Alembert criterion he converges, and according to Weierstrass criterion the functional series (5) converges absolutely and uniformly to the function $w^*(z)$. Thus, the functional series (5) converges. It follows that to the same limit function converges a series of its partial sums, the nth partial sum is nth iteration $w^{[n]}(z)$. Hence $\lim_{n\to\infty}w^{[n]}(z)=w^*(z)$. From the definition of a sequence of iterations (4) (3), taking limit as $n\to\infty$ we obtain

$$\lim_{n\to\infty}w^{[n]}(z)=w_0+\int_0^z a(z)\lim_{n\to\infty}w^{[n-1]}(z)dz\,,n=1,2,3,...$$

i.e. $w^*(z)=w_0+\int_0^z a(z)w^*(z)dz\,,n=1,2,3,...$.

Therefore, $w^*(z)$ is the solution of the integral equation (2), and because of the equivalence it is also solution of the starting complex differential equation (1). Since, according to Picard's theorem, the limit of a series of iterations $\left\{w^{[n]}(z)\right\},n=1,2,...$ is unique, the solution $w(z)$ is equally unique. This means that $w^*(z)=w(z)$ for all $z\in G$. Based on this, it follows that the solution of complex differential equation (1) has the series-iteration form of multiple integrals, i.e.

$$w(z)=w_0\sum_{k=0}^{+\infty}\underbrace{\int_{z_0}^{z} a(z)\int_{z_0}^{z} a(z)\dots\int_{z_0}^{z} a(z)dz^k}_{k\text{-integrals}}. \qquad (6)$$

Note that the function $w(z)$ is analytic solution of (1) in G.

Complex differential equation of second order. Complex canonical linear homogeneous differential equation of second order

$$\frac{d^2w}{dz^2}+a(z)w(z)=0, \qquad (7)$$

with analytical coefficient $a(z)$, according to Picard-Poincarré principle, has an analytical solution $w(z)$, which is a continuous function and satisfies Cauchy-Riemann conditions. In order to determine the solution of the equation (7) and then the number of zeros and the location of the zero solutions (this is very important in engineering), the equation (7) is solved by the series-iterations method by determining the first integral from the normal

form $\frac{dw}{dz} = c_1 - \int_0^z a(z)w(z)dz$. Hence

$$w(z) = c_1 z + c_2 - \int_0^z\int_0^z a(z)w(z)dz^2, \qquad (8)$$

where the integration constants c_1 and c_2 are complex number. If this solution is normalized, for $w(0)=1, w'(0)=0$ we obtain $c_1 = 1$ and $c_2 = 0$, we got one particular integral $w_1(z)$. For $w(0)=0, w'(0)=1$ we have $c_1 = 0$ and $c_2 = 1$, that is, we obtain another particular integral $w_2(z)$. The integrals $w_1(z)$ and $w_2(z)$ are linearly independent, since, it is well known, a complex second order differential equation has two linearly independent particular solutions. We mark them by

$$w_1(z) = 1 - \int_0^z\int_0^z a(z)w_1(z)dz^2, \qquad (9)$$

$$w_2(z) = z - \int_0^z\int_0^z a(z)w_2(z)dz^2. \qquad (10)$$

If with

$$w_1^{[n]}(z) = 1 - \int_0^z\int_0^z a(z)w_1^{[n-1]}(z)dz^2,\ n = 1,2,.., \qquad (11)$$

$$w_2^{[n]}(z) = z - \int_0^z\int_0^z a(z)w_2^{[n-1]}(z)dz^2,\ n = 1,2,.., \qquad (12)$$

we define a sequences of iterations $\left\{w_1^{[n]}(z)\right\}$ and $\left\{w_2^{[n]}(z)\right\}$, with initial approximations $w_1^{[0]}(z)$, $w_1^{[0]}(0) = w_1(0) = 1$ and $w_2^{[0]}(z)$, $w_2^{[0]}(0) = w_2(0) = 0$, then, completely analogously to the differential equation (1), it can be shown that the equation (7) has two linearly independent particular integrals in the form

$$w_1(z) = \sum_{k=0}^{\infty}(-1)^k \underbrace{\int_0^z\int_0^z a(z)dz^2 \dots \int_0^z\int_0^z a(z)dz^2}_{k-\text{ double integrals}} \qquad (13)$$

$$w_2(z) = z +$$

$$+\sum_{k=1}^{n}(-1)^k \underbrace{\int_0^z\int_0^z a(z)dz^2 \dots \int_0^z\int_0^z a(z)dz^2 \int_0^z\int_0^z za(z)dz^2}_{k-\text{ double integrals}}. \qquad (14)$$

Functions (13) and (14) are the sums of the series-iterations of multiple double integrals. These are analytic solutions of equation (7) in the domain G.

We have shown in [1,2] that these functions are very similar to Euclidian's sine and cosine, naming them oscillatory complex sine and (8) cosine with base $a(z)$, i.e. $w_1(z) = \cos_{a(z)} z$, $w_2(z) = \sin_{a(z)} z$ and we approximated them by formula

$$w_1(z) = \cos_{a(x)} z \approx \cos\left(z\sqrt{a(z)}\right),$$

$$w_2(z) = \sin_{a(z)} z \approx \frac{\sin\left(z\sqrt{a(z)}\right)}{\sqrt{a(z)}}.$$

Further, with $F(z) = z\sqrt{a(z)}$ we denoted a function of the frequency and shown that the zero solutions of $w_1(z)$ and $w_2(z)$ are approximately in solutions of equations $z\sqrt{a(z)} = (2n-1)\frac{\pi}{2}, n = 1,2,...$ for cosine and (9) $z\sqrt{a(z)} = n\pi, n = 0,1,...$ for sine solution.

(10)

EXAMPLES

For complex functions, $w(z) > 0$ and $w(z) < 0$ are not defined, the functions $w(z)$ and $-w(z)$ are somehow equal, they don't have a graph, but they have a common modular surface

$$F(x,y) = |w(z)| = |u(x,y) + iv(x,y)|$$
$$= \sqrt{u^2(x,y) + v^2(x,y)}.$$

It can only be zero if $|w(z)| = 0$. If $w(z) = u(x,y) + iv(x,y)$ is analytical and has a zero, then they are isolated, so $|w(z)| = 0$ implies $u(x,y) = 0$, $v(x,y) = 0$. Then there is a possibility for some partial oscillation, at least some of the parts $u(x,y)$ or $v(x,y)$.

Existence of zeros in a complex area is a substitute for oscillatory in the real case. The importance of zeros of complex function is not only in the replacement of oscillatory, but more so in the fact that the number of zeros is equivalent to the rank growth. This is known from real oscillations (see [3,4,5,6,7]). Namely, the real differential equations of second order $y''(x)+a(x)y=0$, under conditions: $a(x)>0$, $a(x)$ is continuous on $[0,+\infty)$ and satisfies Lipschitz condition, integral $\int_0^\infty a(x)dx$ diverges, defines oscillatory function.

In a canonical complex second order differential equation (7), if $a(z)$ is a polynomial, we can expect n oscillations depending on the degree of polynomial $a(z)$. If $a(z)$ is an exponential function, then there are countless zeroes throughout the z-plane.

In the following examples, by series-iteration method, for some special cases of complex equations of the second order, we have tried to determine the zeroes of the solution. Even if that equation had no solutions we have found a particular integral which had zero for some choices of the integration constants.

Example 1. Consider the canonical complex differential equation

$$\frac{d^2w}{dz^2}+k^2w(z)=0 \tag{15}$$

for which $a(z)=k^2$ is a real constant. By series-iteration method we obtain two particular solutions

$$w_1(z)=1-\int_0^z\int_0^z k^2dz^2+\int_0^z\int_0^z k^2dz^2\int_0^z\int_0^z k^2dz^2-\dots$$

$$=1-\frac{(kz)^2}{2!}+\frac{(kz)^4}{4!}-\frac{(kz)^6}{6!}+\dots=\cos(kz)$$

$$w_2(z)=z-\int_0^z\int_0^z zk^2dz^2+\int_0^z\int_0^z k^2dz^2\int_0^z\int_0^z zk^2dz^2-\dots=$$

$$=\frac{1}{k}\left[(kz)^1-\frac{(kz)^3}{3!}+\frac{(kz)^5}{5!}-\dots\right]=\frac{1}{k}\sin(kz).$$

Hence, according Euler's formula

$$w_1(z)=\cos(kz)=\frac{e^{ikz}+e^{-ikz}}{2}=$$

$$=\frac{1}{2}\left[e^{-ky}(\cos kx+i\sin kx)+e^{ky}(\cos kx-i\sin kx)\right]$$

$$=\cos kx\frac{e^{ky}+e^{-ky}}{2}+i\sin kx\frac{e^{-ky}-e^{ky}}{2}$$

$$=\cos kx\cosh ky-i\sin kx\sinh ky$$

and

$$w_2(z)=\frac{1}{k}\sin(kz)=\frac{1}{k}\frac{e^{ikz}-e^{-ikz}}{2i}$$

$$=\frac{1}{2ki}\left[e^{-ky}(\cos kx+i\sin kx)-e^{ky}(\cos kx-i\sin kx)\right]$$

$$=\frac{1}{ki}\left[\cos kx\frac{e^{-ky}-e^{ky}}{2}+i\sin kx\frac{e^{-ky}+e^{ky}}{2}\right]$$

$$=\frac{1}{k}(\sin kx\cosh ky+i\cos kx\sinh ky)$$

The question here is whether $w_1(z)$ and $w_2(z)$ have zeros? From $w_1(z)=0$ follows $u_1(x,y)=0=v_1(x,y)$, that is, $\cos kx\cosh ky=0$ and $\sin kx\sinh ky=0$. We conclude that solution $w_1(z)$ has only isolated zeros on x-axis and in points $x=\frac{(2n-1)}{k}\frac{\pi}{2}, n=1,2,\dots$, $k=const$. For zeros of second solution $w_2(z)$ should apply, $u_2(x,y)=0=v_2(x,y)$, i.e. $\sin kx\cosh ky=0=\cos kx\sinh ky$. Hence, $x=\frac{n\pi}{k}$, $n=0,1,2,\dots$ and $y=0$. Therefore, the zeros of solution $w_2(z)$ are again on the x-axis and in points $x=\frac{n\pi}{k}, n=0,1,2,\dots$. Obviously, the (15) particular solutions of the canonical complex second order differential equation (7) have no common zeros.

Remark. A very important issue is the zero of the general solution

$$w(z)=c_1w_1(z)+c_2w_2(z)=0.$$

This problem has many possibilities because it depends on integration constants.

For example, let us find the solution of the equation (15) in the form $w(z)=e^{rz}$, where r is the complex constant to be determined. Substituting $\frac{dw}{dz}=re^{rz}$, $\frac{d^2w}{dz^2}=r^2e^{rz}$ in (15) we obtain the characteristic equation $e^{rz}(r^2+k^2)=0$. It follows that $r=\pm ki$. We got a new system of particular integrals

$$w_1^*(z)=e^{ikz}, w_2^*(z)=e^{-ikz}.$$

Since $w_1^*(z)=e^{-ky}(\cos kx+i\sin kx)\neq 0$ and $w_2^*(z)=e^{ky}(\cos kx-i\sin kx)\neq 0$, it follows that

$w_1^*(z)$ and $w_2^*(z)$ have no zeros. However, we can determine a linear combination that would have zeros. From $c_1 e^{ikz} + c_2 e^{-ikz} = 0$, i.e. $e^{2ikz} = -\frac{c_2}{c_1}$, we get $z = \frac{1}{2ik}\ln\left(-\frac{c_2}{c_1}\right)$. Hence, $z=0$ for $c_1 = -c_2$. So, from a linear combination $w^*(z) = c_1 e^{ikz} - c_1 e^{-ikz} = 2ic_1 \sin kz$ we get a solution that obviously has zeros.

Example 2. Let $a(z) = z = x + iy$. For canonical complex differential equation (7), that is from (8), we obtain equation

$$w(z) = c_1 z + c_2 - \int_0^z\int_0^z z w(z)\, dz^2.$$

By choosing the constants c_1 and c_2 from the particular integrals, i.e. solutions (9) and (10), by series-iteration method we obtain

$$w_1(z) = 1 - \int_0^z\int_0^z z\, dz^2 + \int_0^z\int_0^z z dz^2 \int_0^z\int_0^z z dz^2 - \ldots,$$

$$w_2(z) = z - \int_0^z\int_0^z z^2\, dz^2 + \int_0^z\int_0^z z dz^2 \int_0^z\int_0^z z^2\, dz^2 - \ldots.$$

Solving these integrals we get

$$w_1(z) = 1 - \frac{z^3}{3\cdot 2} + \frac{z^6}{(6\cdot 5)(3\cdot 2)} - \frac{z^9}{(9\cdot 8)(6\cdot 5)(3\cdot 2)} + \ldots$$

$$w_2(z) = z - \frac{z^4}{4\cdot 3} + \frac{z^7}{(7\cdot 6)(4\cdot 3)} - \frac{z^{10}}{(10\cdot 9)(7\cdot 6)(4\cdot 3)} + \ldots.$$

We note that in the denominator of fractions we have incomplete factorials. This raises the questions: what are the functions $w_1(z)$ and $w_2(z)$, whether they have anything to do with functions $\sin z, \cos z$, or e^z, e^{iz}, because, our goal is to determine the number of zero and the location of zero solutions. When the series-iterations method can't determine the zeros of the solutions and their locations, we go to another way of solving this equation.

Since the canonical complex differential equation of second order (7) has an analytic solution, its second derivative is also analytic, and then follows

$$\frac{d^2 w}{dz^2} = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2}$$
$$= -xu(x,y) + yv(x,y) - i\big(xv(x,y) + yu(x,y)\big).$$

Hence we obtain real system of partial differential equations of second order

$$\frac{\partial^2 u}{\partial x^2} = -xu(x,y) + yv(x,y),$$
$$\frac{\partial^2 v}{\partial x^2} = -xv(x,y) - yu(x,y)$$

which is not easy to solve. Viewed particular, only for $y=0$, we have ordinary derivates $\frac{\partial^2 u}{\partial x^2} = \frac{d^2 u}{dx^2} = u''(x)$ and $\frac{\partial^2 v}{\partial x^2} = \frac{d^2 v}{dx^2} = v''(x)$, and from here the equations $u''(x) = -xu(x)$ and $v''(x) = -xv(x)$. These equations are identical to the real canonical equation of the second order $y''(x) + xy = 0$, and the solutions obtained by the series-iteration method are

$$y_1(x) = 1 - \frac{x^3}{3\cdot 2} + \frac{x^6}{(6\cdot 5)(3\cdot 2)} - \frac{x^9}{(9\cdot 8)(6\cdot 5)(3\cdot 2)} + \ldots$$

$$y_2(x) = x - \frac{x^4}{4\cdot 3} + \frac{x^7}{(7\cdot 6)(4\cdot 3)} - \frac{x^{10}}{(10\cdot 9)(7\cdot 6)(4\cdot 3)} + \ldots.$$

It is known (see [3,4,5,6,7]) that for $x \ge 0$ these solutions define non-elementary functions $y_1(x) = \cos_x x$, $y_2(x) = \sin_x x$, which are oscillatory according to Sturm's theorems. Zero of $y_1(x)$ and $y_2(x)$ are respectively in the solutions of the equations $x\sqrt{x} = (2n-1)\frac{\pi}{2}, n = 1,2,\ldots$ and $x\sqrt{x} = n\pi$, $n = 0,1,2,\ldots$ Since $a(x) = x \to +\infty$, when x grows, then the sine and cosine graphs with a base x have a Prodi character.

Example 3. Let $a(z) = e^z$. If we solve equation (7) by iterations, we have for a particular integral

$$w_1(z) = \cos_{e^z} z = 1 - \int_0^z\int_0^z e^z dz^2 + \int_0^z\int_0^z e^z dz^2 \int_0^z\int_0^z e^z dz^2 - \ldots$$
$$= 1 - \frac{e^z}{1!} + \frac{e^{2z}}{2^2} - \frac{e^{3z}}{2^2 3^2} - \frac{e^{4z}}{2^2 3^2 4^2} + \ldots.$$

We see that this series converges rapidly because $(n!)^2$ is in the denominator. As we can no longer conclude, we solve the differential equation in other way. Substituting $w''(z)=\frac{d^2u}{dx^2}+i\frac{d^2v}{dx^2}$, $a(z)=e^{x+iy}$ and $w(z)=u(x,y)+iv(x,y)$ in (7), we obtain system of equations

$$\frac{d^2u}{dx^2}+e^x\left[u(x,y)\cos y-v(x,y)\sin y\right]=0,$$
$$\frac{d^2v}{dx^2}+e^x\left[u(x,y)\sin y+v(x,y)\cos y\right]=0.$$

Hence, for $y=0$ we get

$$\frac{d^2u}{dx^2}+e^xu(x,y)=0,\ \frac{d^2v}{dx^2}+e^xv(x,y)=0.$$

Since partial derivations are ordinary, we have mathematically identical ordinary canonical differential equations of second order for $u(x)$ and $v(x)$. The coefficient e^x, which grows rapidly, is continuous, and from the first equation of the system, by the series-iterative method for particular integrals

$$u_1(x)=1-\int_0^x\int_0^x e^xdx^2+\int_0^x\int_0^x e^xdx^2\int_0^x\int_0^x e^xdx^2-\ldots,$$
$$u_2(x)=x-\int_0^x\int_0^x xe^xdx^2+\int_0^x\int_0^x e^xdx^2\int_0^x\int_0^x xe^xdx^2-\ldots,$$

we obtain frequency oscillations. For functions $u_1(x)$ and $u_2(x)$ approximate formulas apply

$$u_1(x)=\cos_{e^x}x\approx\cos\sqrt{e^x},$$
$$u_2(x)=\sin_{e^x}x\approx\frac{\sin x\sqrt{e^x}}{\sqrt{e^x}}.$$

It follows that the zero of oscillations are in the solutions of the equations $x\sqrt{e^x}=(2n-1)\frac{\pi}{2}$, $n=1,2,...$ and $x\sqrt{e^x}=n\pi$, $n=0,1,\ldots$.

CONCLUSION

The purpose of learning the theory of differential equations is to solve practical problems in which differential equations are used. A good researcher or scientist is more interested in mathematical modeling of a practical problem, i.e., using the easiest methods for solving and finding solutions of equations.

We believe that the idea of further elaboration of our method is very useful and can be purposefully applied to the case of complex non-homogeneous linear, homogeneous linear higher order differential equations with oscillatory solutions, then on Bernoulli, Riccati etc. Based on our knowledge, the problem of solving differential equations for which the coefficients in normal form are interrupted remains open. All these are ideas that we will deal with in the future.

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