REFINEMENT STABILITY OF CUBIC PYRAMIDAL MESHES

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Abstract
Various elliptic boundary value problems have been used for mathematical models of many engineering problems. Some applied problems require the domain of interest to be divided into two or more separate subdomains, which are triangulated independently. On the other hand, mixed finite element meshes have been used to solve partial differential equations in curved domains. In this case, hexahedral finite elements have been used in the interior of the domain and curved tetrahedral elements have been located in the boundary layer. Additionally, mixed meshes with transitional elements are necessary for the h-p version of the finite element method. Therefore, hybrid meshes have been objects of great interest in the last decades. There are two major requirements that we have to satisfy creating hybrid finite element meshes. The first one is the whole triangulation of the domain of interest to be conforming. On the other hand, we need a stable refinement strategy with as small as a possible number of congruence classes, the lowest measure of degeneracy, and a simple refinement tree. Usually, the pyramidal elements have been used for transitional elements in the interface subdomains. But the homogenous triangulations based on pyramidal elements can be successfully applied for solving elliptic partial differential equations. This paper is devoted to refinement strategies for pyramidal meshes. Several pyramidal elements are tested and a case of instability is demonstrated.

Keywords: hybrid finite element meshes, cubic pyramidal elements, stable refinement strategy, refinement tree, measure of degeneracy.

INTRODUCTION

The three-dimensional hybrid meshes have been widely used by a lot of researchers in the last decades [1,2,3]. Several papers deal with mixed three-dimensional meshes [4,5,6]. But just a few papers are devoted to the properties of cubic pyramidal elements [7]. Unlike the three-dimensional case in the four-dimensional one, the cubic pyramids cannot be coupled conformingly with simplicial elements since the lateral facets of any cubic pyramid are square pyramids. The latter means that the coupling between tesseract meshes and simplicial partitions needs the development of new elements, which have not been in use up to now. Despite this, the cubic pyramidal elements are necessary when tesseract dominant meshes have to be constructed. In this case, the pyramidal elements are located in the boundary layer. The optimal hybrid meshes regarding the wide spectrum of engineering applications are provided with:

- conforming coupling of the adjacent elements;
- stability of the sequence of successive hierarchical triangulations;
- as small as possible degeneracy measure for the sequence of successive hierarchical triangulations;
- as small as possible congruence classes.

Definition 1 Each n-dimensional hypercube can be divided into $2n$ $(n-1)$-hypercubic pyramids. We call these pyramids canonical.

Definition 2 *A pyramid with an identical length of all edges is said to be super regular.*

We define the class $[\hat{R}]$ of the four-dimensional super regular pyramids by

$$\hat{R}[r_1(0,0,0,0), r_2(2,0,0,0), r_3(0,2,0,0), r_4(0,0,2,0), r_5(2,2,0,0), r_6(0,2,2,0),$$

$r_7(2,0,2,0), r_8(2,2,2,0), r_9(1,1,1,1)]$.

Definition 3 *Let* $\underline{v}_i$ $i = 1,2,3,4$ *be four linearly independent vectors. The sum*

$$K = \sum_{i=1}^{4} \alpha_i \underline{v}_i, \quad 0 \le \alpha_i \le 1,$$

is called a four-dimensional parallelotope. The parallelotope is straight if the vectors $\underline{v}_i$ *are mutually perpendicular.*

Further, we present the parallelotope K by

$$K = < \underline{v}_i \mid i = 1,2,3,4 >.$$

Naturally, the following questions arise:

- How to refine the pyramidal elements?
- Are the proposed refinement strategies stable?

This paper deals with answers to these questions. We present a refinement strategy concerning the cubic pyramidal meshes and discuss important cases of stability of the partition method. The proposed method is optimal for all canonical domains [8] since the cubic canonical pyramid is super regular. This phenomenon is only valid in the four-dimensional case [9]. We illustrate by an example that in the general case the proposed partition method is unstable. The results are illustrated by refinement trees and measures of degeneracies.

STABILITY OF PYRAMIDAL MESH SUCCESSIVE REFINEMENTS

In this section, we consider various pyramidal meshes designed to triangulate certain polytopes. Let

$$P[p_i, i = 1,2,\dots,9]$$

be a cubic pyramid and $p_i, i = 10,11,\dots,36$ be the nodes in the middle of the edges as follows:

$$p_{10} = \frac{p_1+p_2}{2}, p_{11} = \frac{p_1+p_3}{2}, p_{12} = \frac{p_1+p_4}{2},$$
$$p_{13} = \frac{p_1+p_5}{2}, p_{14} = \frac{p_1+p_7}{2}, p_{15} = \frac{p_1+p_8}{2},$$
$$p_{16} = \frac{p_1+p_9}{2}, p_{17} = \frac{p_2+p_5}{2}, p_{18} = \frac{p_2+p_7}{2},$$
$$p_{19} = \frac{p_2+p_8}{2}, p_{20} = \frac{p_2+p_9}{2}, p_{21} = \frac{p_3+p_4}{2},$$
$$p_{22} = \frac{p_3+p_5}{2}, p_{23} = \frac{p_3+p_6}{2}, p_{24} = \frac{p_3+p_8}{2},$$
$$p_{25} = \frac{p_3+p_9}{2}, p_{26} = \frac{p_4+p_6}{2}, p_{27} = \frac{p_4+p_7}{2},$$
$$p_{28} = \frac{p_4+p_8}{2}, p_{29} = \frac{p_4+p_9}{2}, p_{30} = \frac{p_5+p_8}{2},$$
$$p_{31} = \frac{p_5+p_9}{2}, p_{32} = \frac{p_6+p_8}{2}, p_{33} = \frac{p_6+p_9}{2},$$
$$p_{34} = \frac{p_7+p_8}{2}, p_{35} = \frac{p_7+p_9}{2}, \ p_{36} = \frac{p_8+p_9}{2}.$$

The section is devoted to the partitioning operator

$$\mathcal{D}P = \{\{P_1[p_1, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}, p_{21}],$$
$$P_2[p_2, p_{10}, p_{13}, p_{14}, p_{15}, p_{17}, p_{18}, p_{19}, p_{20}],$$
$$P_3[p_4, p_{12}, p_{14}, p_{15}, p_{21}, p_{26}, p_{27}, p_{28}, p_{29}],$$
$$P_4[p_7, p_{14}, p_{15}, p_{18}, p_{19}, p_{27}, p_{28}, p_{34}, p_{35}],$$
$$P_5[p_3, p_{11}, p_{13}, p_{15}, p_{21}, p_{22}, p_{23}, p_{24}, p_{25}],$$
$$P_6[p_5, p_{13}, p_{15}, p_{17}, p_{19}, p_{22}, p_{24}, p_{30}, p_{31}],$$
$$P_7[p_6, p_{15}, p_{21}, p_{23}, p_{24}, p_{26}, p_{28}, p_{32}, p_{33}],$$
$$P_8[p_8, p_{15}, p_{19}, p_{24}, p_{28}, p_{30}, p_{32}, p_{34}, p_{36}],$$
$$P_9[p_9, p_{16}, p_{20}, p_{25}, p_{29}, p_{31}, p_{33}, p_{35}, p_{36}],$$
$$P_{10}[p_{15}, p_{16}, p_{20}, p_{25}, p_{29}, p_{31}, p_{33}, p_{35}, p_{36}]\},$$
$$\{W_1[p_{15}, p_{21}, p_{23}, p_{24}, p_{25}, p_{33}],$$
$$W_2[p_{11}, p_{13}, p_{15}, p_{16}, p_{21}, p_{25}],$$
$$W_3[p_{12}, p_{14}, p_{15}, p_{16}, p_{21}, p_{29}],$$
$$W_4[p_{15}, p_{21}, p_{26}, p_{28}, p_{29}, p_{33}],$$
$$W_5[p_{15}, p_{24}, p_{28}, p_{32}, p_{33}, p_{36}],$$
$$W_6[p_{14}, p_{15}, p_{27}, p_{28}, p_{29}, p_{35}],$$
$$W_7[p_{15}, p_{19}, p_{28}, p_{34}, p_{35}, p_{36}],$$
$$W_8[p_{13}, p_{15}, p_{17}, p_{19}, p_{20}, p_{31}],$$
$$W_9[p_{15}, p_{19}, p_{24}, p_{30}, p_{31}, p_{36}],$$
$$W_{10}[p_{14}, p_{15}, p_{18}, p_{19}, p_{20}, p_{35}],$$
$$W_{11}[p_{13}, p_{15}, p_{22}, p_{24}, p_{25}, p_{31}],$$
$$W_{12}[p_{10}, p_{13}, p_{14}, p_{15}, p_{16}, p_{20}],$$
$$W_{13}[p_{15}, p_{19}, p_{20}, p_{31}, p_{35}, p_{36}],$$
$$W_{14}[p_{15}, p_{28}, p_{29}, p_{33}, p_{35}, p_{36}],$$
$$W_{15}[p_{15}, p_{16}, p_{21}, p_{25}, p_{29}, p_{33}],$$
$$W_{16}[p_{13}, p_{15}, p_{16}, p_{20}, p_{25}, p_{31}],$$
$$W_{17}[p_{15}, p_{24}, p_{25}, p_{31}, p_{33}, p_{36}],$$
$$W_{18}[p_{14}, p_{15}, p_{16}, p_{20}, p_{29}, p_{35}]\}\}.$$

The refinement strategy related to the partition operator $\mathcal{D}$ is a generalization of the subdivision procedure described by Ainsworth and Fu [10]. Here, we emphasize the fact that the wedges play the role of the auxiliary tetrahedra in the three-dimensional case. The operator $\mathcal{D}$ divides any cubic pyramid into ten pyramids and eighteen wedges. The wedges in the four-dimensional cases are bipentatopes. On the next level, we refine each wedge

$$W[w_i, i = 1,2,\dots,9]$$

into wedges by the following subdivision operator

$$\mathcal{B}W = \{W_1[w_5, w_{10}, w_{13}, w_{16}, w_{18}, w_{20}],$$
$$W_2[w_6, w_{11}, w_{14}, w_{17}, w_{19}, w_{20}],$$
$$W_3[w_8, w_{10}, w_{13}, w_{16}, w_{18}, w_{20}],$$
$$W_4[w_8, w_{11}, w_{14}, w_{17}, w_{19}, w_{20}],$$
$$W_5[w_3, w_8, w_{12}, w_{15}, w_{16}, w_{17}],$$
$$W_6[w_4, w_8, w_9, w_{15}, w_{18}, w_{19}],$$
$$W_7[w_1, w_7, w_8, w_9, w_{10}, w_{11}],$$

$$W_8[w_2, w_7, w_8, w_{12}, w_{13}, w_{14}],$$
$$W_9[w_8, w_{15}, w_{16}, w_{17}, w_{18}, w_{19}],$$
$$W_{10}[w_8, w_{12}, w_{13}, w_{14}, w_{16}, w_{17}],$$
$$W_{11}[w_7, w_8, w_{10}, w_{11}, w_{13}, w_{14}],$$
$$W_{12}[w_8, w_9, w_{10}, w_{11}, w_{18}, w_{19}],$$
$$W_{13}[w_8, w_{16}, w_{17}, w_{18}, w_{19}, w_{20}],$$
$$W_{14}[w_8, w_{13}, w_{14}, w_{16}, w_{17}, w_{20}],$$
$$W_{15}[w_8, w_{10}, w_{11}, w_{18}, w_{19}, w_{20}],$$
$$W_{16}[w_8, w_{10}, w_{11}, w_{13}, w_{14}, w_{20}]\}\},$$

where:

$$w_7 = \frac{w_1+w_2}{2},\ w_8 = \frac{w_1+w_3}{2},$$
$$w_9 = \frac{w_1+w_4}{2},\ w_{10} = \frac{w_1+w_5}{2},$$
$$w_{11} = \frac{w_1+w_6}{2},\ w_{12} = \frac{w_2+w_3}{2},$$
$$w_{13} = \frac{w_2+w_5}{2}, w_{14} = \frac{w_2+w_6}{2},$$
$$w_{15} = \frac{w_3+w_4}{2}, w_{16} = \frac{w_3+w_5}{2},$$
$$w_{17} = \frac{w_3+w_6}{2}, w_{18} = \frac{w_4+w_5}{2},$$
$$w_{19} = \frac{w_4+w_6}{2}, w_{20} = \frac{w_5+w_6}{2}.$$

The superposition $\mathcal{L} = \mathcal{D}^{\circ}\mathcal{B}$ can be used for partitioning of an arbitrary parallelotope and of course for subdivision of any canonical domain.

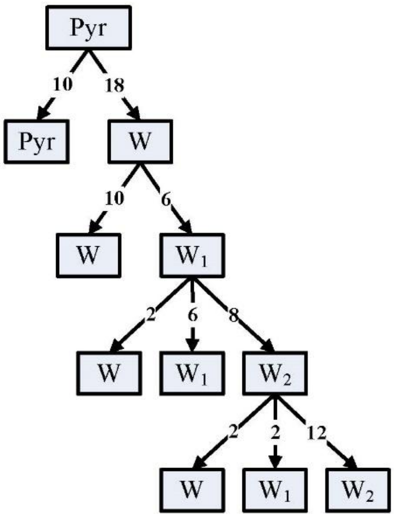

Fig. 1. *The refinement tree generated by $\mathcal{L}\hat{T}$.*

The class $[\hat{R}]$ has been analyzed by Petrov et al. in [9]. Since each tesseract can be divided into eight super regular pyramids, these elements are applicable for triangulating canonical domains. The elements of the class $[R]$ are provided with the optimal two-level refinement tree and the optimal number of congruence classes in the four-dimensional case [9]. Some authors have been used the cubic pyramid

$$\hat{T} = [\hat{t}_1(-1,-1,-1,-1),$$
$$\hat{t}_2(-1,1,1,-1), \hat{t}_3(-1,-1,1,-1),$$
$$\hat{t}_4(1,1,-1,-1), \hat{t}_5(-1,1,-1,-1),$$
$$\hat{t}_6(1,-1,1,-1), \hat{t}_7(1,-1,-1,-1),$$
$$\hat{t}_8(1,1,1,-1), \hat{t}_9(0,0,0,1)]$$

for the reference element. The pyramid $\hat{T}$ has real advantages with respect to the symmetry groups related to quadrature formulae. Unfortunately, this pyramid is regular but not super regular. The successive applications of the operator $\mathcal{L}$ on the pyramidal elements from the class $\hat{T}$ generate a five-level refinement tree and four congruence classes, see Figure 1. This result is true for all regular pyramids, which do not belong to the class $[\hat{R}]$.

Let the parallelotope

$$K =< \underline{v}_i \mid i = 1,2,3,4 >$$

satisfies

$$\frac{\|\underline{v}_i\|}{\|\underline{v}_j\|} \neq \frac{\|\underline{v}_k\|}{\|\underline{v}_l\|}, \qquad (1)$$

where (i,j,k,l) is a permutation of the numbers 1,2,3,4 and $\|\cdot\|$ is the Euclidean norm in $\boldsymbol{R}^4$. Then the number of classes of similarity grows up as it is shown in Figure 2. We illustrate the results in Figure 2 with the next example.

Example 1 In this example, we consider the parallelotope

$$\hat{K} =< \underline{v}_1(e,0,0,0), \underline{v}_2(0,\varphi,0,0),$$
$$\underline{v}_3(0,0,\sqrt{3},0), \underline{v}_4(0,0,0,2\pi) >,$$

where e is Euler's number and φ is the golden ratio. Obviously, $\hat{K}$ satisfies the negation (1). The parallelotope $\hat{K}$ is divided into eight pyramids from the class $[\tilde{P}]$, where

$$\tilde{P}[\tilde{p}_1(0,0,0,0), \tilde{p}_2(e,0,0,0),$$
$$\tilde{p}_3(0,\varphi,0,0), \tilde{p}_4(0,0,\sqrt{3},0), \tilde{p}_5(e,\varphi,0,0),$$
$$\tilde{p}_6(0,\varphi,\sqrt{3},0), \tilde{p}_7(e,0,\sqrt{3},0),$$
$$\tilde{p}_8(e,\varphi,\sqrt{3},0), \tilde{p}_9\left(\frac{e}{2},\frac{\varphi}{2},\frac{\sqrt{3}}{2},\pi\right)\Bigg].$$

The results for the degeneracy measure in the first four levels are presented in Table 1, Table 2, and Table 3

Pyr	W_1	W_2	W_3
2.55421	3.69252	4.05123	4.12243

Table 1 *The degeneracy measure for the cubic pyramid $\tilde{P}$ and the first three wedge successors.*

W_4	W_5	W_6	W_7
6.92876	8.57064	8.92648	8.07537

***Table 2** The degeneracy measure for the next four wedge successors.*

W_8	W_9	W_{10}	W_{11}	W_{12}
8.22481	9.12830	9.52352	10.1834	10.4365

***Table 3** The degeneracy measure for the last five wedge successors.*

Let τ_0 be the initial conforming triangulation of the parallelotope $\widehat{K}$ by eight cubic pyramidal elements and $\tau_k = \mathcal{L}^k\widehat{K}\ k = 1,2, \ldots$ are the triangulations in the next levels. By applying the results in Table 1, Table 2, Table 3, and the refinement tree in Figure 2, we have

$$\delta(\tau_0) = 2.55421,\ \ \delta(\tau_1) = 4.12243,$$
$$\delta(\tau_2) = 8.92648,\ \delta(\tau_3) = 10.4365.$$

Moreover,

$$\mu(\tau_0) = 1,\ \ \mu(\tau_1) = 4,$$
$$\mu(\tau_2) = 7,\ \mu(\tau_3) = 13,$$

where $\mu(\tau_k)$ is the number of the congruence classes in the triangulation τ_k. These results indicate that $\mathcal{L}$ generates an unstable sequence $\{\tau_k\}$ of successive triangulations.

CONCLUSION

The four-dimensional hybrid meshes are the object of interest in this paper. The paper describes a partition method, which is optimal concerning the number of congruence classes for all canonical domains. The partition method remains stable for all regular pyramidal elements. The simplest refinement tree is obtained in the case of canonical domains. The proposed refinement strategy is tested on the parallelotope $\widehat{K}$ satisfying the condition (1). The refinement of $\widehat{K}$ is a critical example of the instability of the partitioning operator $\mathcal{L}$ in the general case. The latter means that the interface subdomain should be canonical or domain, which can be triangulated by regular pyramids at least. Such a requirement is not very restrictive from the computational point of view.

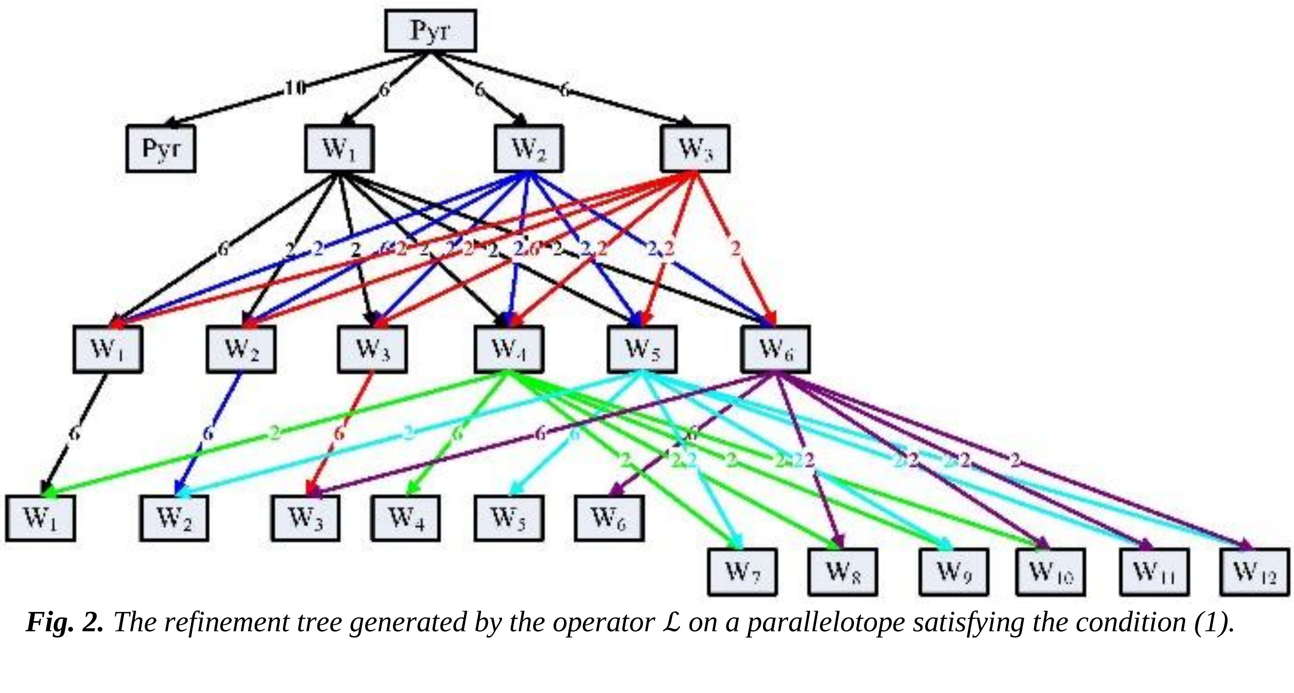

***Fig. 2.** The refinement tree generated by the operator $\mathcal{L}$ on a parallelotope satisfying the condition (1).*

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