SOME COMMON FIXED POINT THEOREMS IN C^*-ALGEBRA VALUED METRIC SPACES

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Abstract

In this paper, we proved some unique common fixed point results in setting of C^*-algebra-valued metric spaces for two pairs of weakly compatible mappings, satisfying a contractive condition given by Pachpatte [Pachpatte, B. G. Fixed point theorems satisfying new contractive type conditions, Soochow Journal of Mathematics. 16(2), pp.173-183, (1990)] using property (E.A).

Keywords: C^*-algebra, C^*-algebra-valued metric spaces, Pachpatte contractive condition, property (E.A), weakly compatible mappings.

INTRODUCTION

In Ma et al. [10] introduced C^*-algebra-valued metric spaces as a new concept which are more general than metric space, replacing the set of real numbers by C^*- algebras, and establish a fixed point theorem for self-maps with contractive or expansive conditions on such spaces, analogous to the Banach contraction principle.

A $*$-algebra is a complex algebra with linear involution $*$ such that $y^{**}=y$ and $(yz)^*=z^*y^*$, for any $y,z\in E$. If $*$- algebra together with complete sub multiplicative norm satisfying $\|y^*\|=\|y\|$ for all $y\in E$, then $*$- algebra is said to be a Banach $*$-algebra. A C^*- algebra is a Banach $*$-algebra such that $\|y^*y\|=\|y\|^2$ for all $y\in E$.

If a normed algebra E admits a unit 1_E, $a1_E=1_E a=a$ for all $a\in E$, $\|1_E\|=1$, then we say that E is a unital normed algebra. A complete unital normed algebra E is called unital Banach algebra.

A positive element of E is an element $a\in E$ such that $a^*=a$ and its spectrum $\sigma(a)\subset R_+$ where $\sigma(a)=\{\lambda\in R:\lambda 1_E-a$ is non-invertible$\}$. The set of all positive elements will be denoted by E_+. Such elements allow us to define a partial ordering $\succeq$ on the elements of E. That is, $b\succeq a$ if and only if $b-a\in E_+$. If $a\in E$ is positive, then we write $a\succeq 0_E$, where 0_E is the zero element of E. Each positive element a of a C^*-algebra E has a unique positive square root. From now on, by E we mean a unital C^*-algebra with identity element 1_E. The sum of two positive elements in a C^*-algebra is a positive element. If a is an arbitrary element of a C^*-algebra E, then a^*a is positive. Let E be a C^*- algebra and if $a,b\in E^+$ such that $a\preceq b$, then for any $x\in E$, both x^*ax and x^*bx are positive elements and $x^*ax\preceq x^*bx$.

Further, $E_+=\{a\in E: a\succeq 0_E\}$ and $(a^*a)^{\frac{1}{2}}=|a|$.

Very simply, E has an algebraic structure and a topological structure coming from a norm. The condition that E be a Banach algebra expresses a compatibility between these structures. All over this paper, E means a unital C*-algebra with a unit I, R is set of real numbers and R^+ is the set of non-negative real numbers and $M_n(R)$ is $n \times n$ matrix with real entries R.

Lemma 1. [9] Suppose that E is a unital C*-algebra with a unit 1_E. For any $x \in E_+$, we have $x \preceq 1_E$, if and only if $\|x\| \le 1$. If $a \in E_+$, with $\|a\| < \frac{1}{2}$ then $1_E - a$ is invertible and $\|a(1_E - a)^{-1}\| < 1$. Suppose that $a, b \in E$ with $a, b \succeq 0_E$, and $ab = ba$, then $ab \succeq 0_E$. By E' we denote the set

$$\{a \in E : ab = ba, \text{ for all } b \in E\}.$$

Let $a \in E'$ if $b, c \in E$ with $b \succeq c \succeq 0_E$, and $1_E - a \in E'$ is an invertible operator, then $(1_E - a)^{-1} b \succeq (1_E - a)^{-1} c$.

Based on the concept and properties of C*-algebras, recently in Ma et al. [10] introduced the concept of C*-algebra-valued metric spaces as a new concept which are more general than metric space, replacing the set of real numbers by C*-algebras as follows:

Definition 1. [10] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ is defined, with the following properties:

(1.1) $0_E \preceq d(x, y)$ for all x and y in X,

(1.2) $d(x, y) = 0_E$ if and only if $x = y$,

(1.3) $d(x, y) = d(y, x)$ for all x and y in X,

(1.4) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is said to be a C*-algebra-valued metric on X, and (X, E, d) is said to be a C*-algebra-valued metric space.

Definition 2. [10] Suppose that (X, E) is a C*- algebra -valued metric space. A mapping T:$X \to X$ is called C*-algebra-valued contractive mapping on X, if there is an $P \in E$ with $\|P\| < 1$ such that $d(Tx, Ty) \preceq P^*(d(x, y))P$ for all $x, y \in X$.

Example 1. [10] Let $X = R$ and $E = M_2(R)$. Defined $(x, y) = diag(|x - y|, \alpha|x - y|)$, where $x, y \in R$ and $\alpha \ge 0$ is a constant. Then d is a C*-algebra-valued metric and $(X, M_2(R), d)$ is a complete C*-algebra-valued metric space by the completeness of R.

Definition 3. [10] Let (X, E, d) is a C*-algebra valued metric space and let $\{x_n\}$ be a sequence in X. If

(1.5) for any $\varepsilon > 0$, there is N such that for all $n > N$, $\|d(x_n, x)\| \le \varepsilon$, then the sequence $\{x_n\}$ is said to be convergent, and we denote it as $\lim_{n\to\infty} x_n = x$.

(1.6) for any $\varepsilon > 0$, there is N such that for all $m,n > N$, $\|d(x_m, x_n)\| \le \varepsilon$, then the sequence $\{x_n\}$ is said to be Cauchy sequence.

(1.7) C*-algebra valued metric space is said to be complete if every Cauchy sequence in X with respect to E is convergent.

For more details, one can refers ([2], [3], [5], [6], [7], [8], [12], [13], [14]).

Definition 4. [4] A pair of self-mappings A,B: $X \to X$ is called weakly compatible if they commute at their coincidence point, that is, if there is a point $z \in X$ such that $Az = Bz$, then $ABz = BAz$, for each $z \in X$.

The definition of weakly compatible is used in similar mode in C*-algebra-valued metric space as in metric spaces.

The definition of property (E.A) has been introduced in [1] as follows:

Definition 5. [1] Let A,B: $X \to X$ be two self-mappings of a metric space (X, d). The pair (A,B) is said to satisfy property (E.A)**,** if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(Ax_n, u) = \lim_{n\to\infty} d(Bx_n, u) = 0,$$

for some $u \in X$.

Now we define property (E.A) in C*-algebra-valued metric spaces as follows:

Definition 6. Let A,B: $X \to X$ be two self-mappings of a metric space (X, d). The pair (A,B) is said to satisfy property (E.A), if there exists a sequence $\{x_n\}$ in such that

$$\lim_{n\to\infty} d(Ax_n, u) = \lim_{n\to\infty} d(Bx_n, u) = 0_E,$$

for some $u \in X$.

MAIN RESULTS

In this section, common fixed point results for the pairs in setting of C*-algebra-valued metric space using weakly compatible and property (E.A), have been proved, by the contractive conditions given by Pachpatte [11] as follows:

Theorem 1. Let (X,E,d) be a C*-algebra-valued metric space and let P,Q,R,S : $X \to X$ be four self-mappings satisfying the following:

(C_1) P(X) ⊆Q(X), R(X) ⊆S(X);

(C_2)

$$[d(Rx,Py)]^3 \preceq A^*(d(Qx,Sy)d(Qx,Rx)d(Sy,Py))A,$$

for all $x,y \in X$, where $\|A\|<1$;

(C_3) the pairs (R,Q) and (P,S) are weakly compatible;

(C_4) one of the pairs (R,Q) and (P,S) satisfies property (E.A).

If the range of one of the mappings S(X) or Q(X) is closed subspace of X, then the mappings P,Q,R, and S have a unique common fixed point in X.

Proof. Suppose that the pair (P,S) satisfies property (E.A). Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Sx_n = z, \tag{1}$$

for some $z \in X$.

Further, since P(X) ⊆ Q(X), there exists a sequence $\{y_n\}$ in X such that $Px_n = Qy_n$. Hence $\lim_{n\to\infty} Qy_n = z$. Now our claim is $\lim_{n\to\infty} Ry_n = z$. Putting $x = y_n$ and $y = x_n$ in condition (C_2), we have

$$[d(Ry_n,Px_n)]^3 \preceq A^*(d(Qy_n,Sx_n)d(Qy_n,Ry_n)d(Sx_n,Px_n))A$$
$$= A^*(d(Px_n,Sx_n)d(Px_n,Ry_n)d(Px_n,Sx_n))A.$$

Which implies that

$$\|d(Ry_n,Px_n)\|^3 \le \|A\|^2\left(\|d(Px_n,Sx_n)\|\|d(Px_n,Ry_n)\|\|d(Px_n,Sx_n)\|\right)$$

By dividing two sides of the above inequality with $\|d(Ry_n,Px_n)\|$, we get

$$\|d(Ry_nPx_n)\|^2 \le \|A\|^2\left(\|d(Px_n,Sx_n)\|^2\right).$$

Taking limit $n \to \infty$, we have

$$\|d(Ry_n,Px_n)\| \le \|A\|\|d(Px_n,Sx_n)\| = 0$$

i.e., $\lim_{n\to\infty} Ry_n = \lim_{n\to\infty} Px_n = z.$

Now, suppose that Q(X) is a closed subspace of X, then there exists $u \in X$ such that $z = Qu$. Subsequently, we have

$$\lim_{n\to\infty} Ry_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Qy_n = z = Qu. \tag{2}$$

We claim that Ru = Qu. Putting $x = u$, $y = x_n$ in (C_2), we get

$$[d(Ru,Px_n)]^3 \preceq A^*(d(Qu,Sx_n)d(Qu,Ru)d(Sx_n,Px_n))A,$$

and letting $n\to\infty$ and using (2) we have

$$\|d(Ru,z)\|^3 \le \|A\|^2\left(\|d(z,z)\|\|d(z,Ru)\|\|d(z,z)\|\right) = 0,$$

and consequently $Ru = z = Qu.$ Thus z is a coincidence point of (R,Q). Since R(X) ⊆S(X), there exists $v \in X$ such that $Ru = Sv.$ Hence $Ru = Qu = Sv = z.$

Now we show that v is a coincidence point of (P,S) that is, $Pv = Sv = z.$ Now putting $x = u$, $y = v$ in (C_2), we get

$$[d(Ru,Pv)]^3 \preceq A^*(d(Qu,Sv)d(Qu,Ru)d(Sv,Pv))A, \quad \text{i.e}$$

$$\|d(z,Pv)\|^3 \le \|A\|^2\left(\|d(z,Sv)\|\|d(z,z)\|\|d(Sv,Pv)\|\right).$$

Thus $Pv = z.$ Hence $Pv = Sv = z$ and v is a coincidence point of P and S. Since the pairs (R,Q) and (P,S) are weakly compatible, and z and v are their coincidence point respectively, so we have RQu = QRu = Rz = Qz, $PSv = SPv = Pz = Sz.$

In order to show that z is a common fixed point of these mappings, on putting $x = u$ and $y = z$ in condition (C_2), we have

$$[d(z,Pz)]^3 = [d(Ru,Pz)]^3 \preceq A^*(d(Qu,Sz)d(Qu,Ru)d(Sz,Pz))A, \quad \text{i.e}$$

$$\|d(z,Pz)\|^3 \le \|A\|^2\left(\|d(Qu,Sz)\|\|d(Qu,Ru)\|\|d(Sz,Pz)\|\right).$$

Hence, $\|d(z,Pz)\|^3 \le 0.$

Thus, $Rz = Qz = Sz = Pz = z.$

Similarly, we can complete the proof for case in which S(X) is closed subspace of X.

Existence

To prove that z is a unique common fixed point, let us suppose that p is another common fixed point of P, Q, R, and S. Putting $x = p$ and $y = z$ in condition (C$_2$), we have

$$[d(p,z)]^3 = [d(Rp,Pz)]^3$$

$$\preceq A^*(d(Qp,Sz)d(Qp,Rp)d(Sz,Pz))A, \quad \text{i.e}$$

$$\|d(p,z)\|^3$$

$$\le \|A\|^2 \left(\|d(Qp,Sz)\|\|d(Qp,Rp)\|\|d(Sz,Pz)\|\right).$$

Hence, $\|d(p,z)\|^3 \le 0$ is a contradiction.

Thus $z = p$. Consequently, Rz = Qz = Pz = Sz = z and z is the unique common fixed point of P,Q,R, and S. Putting S=Q in Theorem 1 we have the following corollary:

Corollary 1. Let P,Q and R be three self-mappings of a C*-algebra-valued metric space (X, E, d) satisfying the inequality

(C$_5$)

$$[d(Rx,Py)]^3$$

$$\preceq A^*(d(Qx,Qy)d(Qx,Rx)d(Qy,Py))A,$$

for all $x, y \in X$, where $\|A\| < 1$.

Suppose that the following conditions hold:

(C$_6$) Q(X) ⊇R(X)∪P(X),

(C$_7$) both the pairs (Q,R) and (Q,P) are weakly compatible,

(C$_8$) one of the pairs (Q,R) and (Q,P) satisfies the property (E.A).

If Q(X) is closed subspace of X, then R, P, and Q have a unique common fixed point in X. In Theorem 1 if we put R = P and Q = S, we have the following.

Corollary 2. Let (X, E, d) be a C*-algebra-valued metric space and let R and P be two self-mappings satisfying the following:

(C$_9$) R(X) ⊆Q(X);

(C$_{10}$)

$$[d(Rx,Py)]^3$$

$$\preceq A^*(d(Qx,Qy)d(Qx,Rx)d(Qy,Ry))A,$$

for all $x, y \in X$, where $\|A\| < 1$,

(C$_{11}$) (Q, R) is a weakly compatible pair;

(C$_{12}$) the pair (Q, R) satisfies property (E.A). If Q(X) is closed subspace of X, then Q and R have the unique common fixed point in X.

Theorem 2. Let P, Q, R, and S be four self-mappings of a C*-algebra-valued metric space (X, E, d) satisfying the following:

(C$_{13}$) P(X) ⊆Q(X) and R(X) ⊆S(X);

(C$_{14}$)

$$d(Rx,Py) \preceq$$

$$A^* \left(\max \left\{ \begin{matrix} (d(Qx,Sy))^2, (d(Qx,Rx))^2, (d(Sy,Py))^2, \\ \frac{1}{2}(d(Qx,Py))^2, \frac{1}{2}(d(Sy,Rx))^2 \end{matrix} \right\} \times \right.$$

$$\left. (d(Qx,Rx) + d(Sy,Py))^{-1} \right) A.$$

If $d(Qx,Rx) + d(Sy,Py) \ne 0_E$ where $\|A\| <$, or $d(Rx,Py) = 0_E$ if $d(Qx,Rx) + d(Sy,Py) = 0_E$, for all $x, y \in X$,

(C$_{15}$) the pairs (R,Q) and (P,S) are weakly compatible;

(C$_{16}$) one of the pairs (R,Q) and (P,S) satisfies property (E.A).

If the range of one of the mappings S(X) or Q(X) is a closed subspace of X, then the mappings P, Q, R, and S have a unique common fixed point in X.

Proof. Let us suppose that $d(Qx,Rx) + d(Sy,Py) \ne 0_E$ so $d(Rx,Py) \ne 0_E$ and the pair (P,S) satisfies property (E.A). Then there exists a sequence $\{x_n\}$ in such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Sx_n = z,$ for some $z \in X$. Since P(X) ⊆Q(X), there exists a sequence $\{y_n\}$ in X such that $Px_n = Qy_n$. Hence $\lim_{n\to\infty} Qy_n = z.$ Next we claim that $\lim_{n\to\infty} Ry_n = z.$ In inequality (C$_{14}$), putting $x = y_n$ and $y = x_n$, we get

$$d(Ry_n, Px_n) \preceq$$

$$A^* \left(\max \left\{ \begin{matrix} (d(Qy_n,Sx_n))^2, (d(Qy_n,Ry_n))^2, (d(Sx_n,Px_n))^2, \\ \frac{1}{2}(d(Qy_n,Px_n))^2, \frac{1}{2}(d(Sx_n,Ry_n))^2 \end{matrix} \right\} \times \right.$$

$$\left. (d(Qy_n,Ry_n) + d(Sx_n,Px_n))^{-1} \right) A.$$

$$= A^*\left(\max\left\{ \begin{matrix} (d(Px_n, Sx_n))^2, (d(Px_n, Ry_n))^2, 0_A, \\ \frac{1}{2}(d(Sx_n, Ry_n))^2 \end{matrix} \right\} \times (d(Px_n, Ry_n) + d(Sx_n, Px_n))^{-1} \right) A.$$

Hence,

$$\|d(Ry_n, Px_n\| \le \|A\|^2 \left(\left\| \max\left\{ \begin{matrix} (d(Px_n, Sx_n))^2, (d(Px_n, Ry_n))^2, \\ 0, \frac{1}{2}(d(Sx_n, Ry_n))^2 \end{matrix} \right\} \times (d(Px_n, Ry_n) + d(Sx_n, Px_n))^{-1} \right\| \right)$$

and letting $n \to \infty$, we have

$$(1 - \|A\|^2)\|d(Ry_n, Px_n)\| \le 0$$

which is a contradiction, since $\|A\| < 1$. Therefore, $\lim_{n\to\infty} Ry_n = \lim_{n\to\infty} Px_n = z$. Assuming Q($X$) is closed subspace of X, then z=Qu for some $u \in X$. Now, we obtain

$$\lim_{n\to\infty} Ry_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Qy_n = z = Qu. \quad (3)$$

Our aim is to prove Ru = Q and for this putting $x = u$ and $y = x_n$ in (C_{14}) we get

$$d(Ru, Px_n) \preceq A^*\left(\max\left\{ \begin{matrix} (d(Qu, Sx_n))^2, (d(Qu, Ru))^2, (d(Sx_n, Px_n))^2, \\ \frac{1}{2}(d(Qu, Px_n))^2, \frac{1}{2}(d(Sx_n, Ru))^2 \end{matrix} \right\} \times (d(Qu, Ru) + d(Sx_n, Px_n))^{-1} \right) A.$$

Letting $n \to \infty$ and using (3) we get $(1 - \|A\|^2)\|d(Ru, z)\| \le 0$, and R$u$ = z since $\|A\| < 1$. Therefore, u is a coincidence point of (R, Q). Weak compatibility of the pair (R,Q) implies that RQu=QRu=Rz=Qz. Otherwise, since R(X) ⊆S(X), there exists $v \in X$ such that $Ru = Sv$.

Hence, Ru = Qu = Sv = z. To show that v is a coincidence point of pair (P,S), by using similar arguments in Theorem 1 and inequality (C_{14}) we have

$$d(Ru, Pv) \preceq A^*\left(\max\left\{ \begin{matrix} (d(Qu, Sv))^2, (d(Qu, Ru))^2, (d(Sv, Pv))^2, \\ \frac{1}{2}(d(Qu, Pv))^2, \frac{1}{2}(d(Sv, Ru))^2 \end{matrix} \right\} \times (d(Qu, Ru) + d(Sv, Pv))^{-1} \right) A.$$

Hence

$$\|d(z, Pv)\| \le \|A\|^2 \|d(z, Pv\|$$

and then $Pv = z$ because $\|A\| < 1$. With the same assertions as in Theorem 1 one gets that z is a common coincidence point of P, Q, R, and S. Other details of Theorem 2 in which z is a unique common fixed point of the mappings P, Q, R, and S can be obtained in view of the final part of the proof of Theorem 1.

Remark 1. We note that the conclusions of Theorem 2 are still valid if we replace inequality (C_{14}) with the following inequality:

$$d(Rx, Py) \preceq A^*\left(\max\left\{ \begin{matrix} \frac{d(Qx, Py)[1 + d(Qx, Sy) + d(qx, Sy)]}{2[1 + d(Qx, Sy)]}, \\ \frac{d(Sy, Rx)[1 + d(Qx, Py) + d(Sy, Py)]}{2[1 + d(Qx, Sy)]} \end{matrix} \right\} \right) A$$

CONCLUSION

Recently, based on the concept and properties of C*-algebras, Ma et al. [10] introduced the concept of C*-algebra-valued metric spaces as a new concept which are more general than metric space replacing the set of real numbers by C*-algebras.

This paper is committed to proving common fixed point results for the pairs in setting of C*-algebra-valued metric space using weakly compatible and property (E.A), by the contractive conditions given by Pachpatte [11].

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