C-CLASS FUNCTIONS WITH COUPLED COINCIDENCE POINT RESULTS FOR A GENERALIZED COMPATIBLE PAIR IN ORDERED b-METRIC SPACES

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Abstract

In this paper, by using the C-class functions and a new approach we present some coupled coincidence point results for a pair of mappings satisfying generalized (ϕ,ψ)-weakly contractive condition in the setting of ordered b-metric spaces.

Keywords: coupled coincidence point, generalized compatibility, mixed monotone property, C-class functions

INTRODUCTION

Guo and Lakshmikantham [9] studied the concept of coupled fixed points, they first give some existence theorems of the coupled fixed points for continuous and discontinuous operators. Later, Bhaskar and Lakshmikantham [5] studied monotone property and supported this by providing an application to the existence of periodic boundary value problems. Lakshmikantham and Ćirić [13] introduce the concept of a mixed g-monotone mapping and prove coupled coincidence and coupled common fixed point theorems for such nonlinear contractive mappings in partially ordered complete metric spaces.

In this paper, we use the notion of C-class of function which is generalization of altering distance function and by using this definition we have improved the results of Hussain, Abbas, Azam and Ahmad [14] in the setting of b-metric space and proved coupled fixed point theorem. For more about fixed point results via C-class functions see [17].

Definition 1. [8] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric if and only if for all $x, y, z \in X$, the following conditions are satisfied:

(b1) $d(x, y) = 0$ if and only if $x = y$,

(b2) $d(x, y) = d(y, x)$,

(b3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

Definition 2. [6] Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called b-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$, as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x.$

Definition 3. [6] Let (X,d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called b-Cauchy if and only if $d(x_n, x_m) \to 0,$ as $n,m \to +\infty.$

Definition 4. [6] The b-metric space (X,d) is b-complete if every b-Cauchy sequence in X b-converges.

Definition 5. Let X be a nonempty set. Then $(X, d, \preceq)$ is called a partially ordered b-metric space if and only if d is a b-metric on a partially ordered set $(X, \preceq)$.

For more on coupled fixed point theory we refer to the reviews (see, [1], [2], [3], [7], [9], [13], [14], [18]). Sevaral papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in b-metric spaces (see [7], [8], [10], [16]).

Definition 6. [5] Let $(X,\preceq)$ be a partially ordered set. The mapping $F: X\times X \to X$ is said to have the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-icreasing in its second argument, that is, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$, for any $y \in X$ and for all $y_1, y_2 \in X$, $y_1 \preceq y_2$ implies $F(x, y_1) \succeq F(x, y_2),$ for any $x \in X$.

Lakshmikantham and Ćirić [13] generalized the concept of a mixed monotone property as follows.

Definition 7. [13] Let $(X,\preceq)$ be a partially ordered set and g a self mapping on X. A mapping $F: X\times X \to X$ is said to have a mixed g-monotone property if for all $x_1, x_2 \in X$, $g x_1 \preceq g x_2$ implies that $F(x_1, y) \preceq F(x_2, y)$, for any $y \in X$ and for all $y_1, y_2 \in X$, $g y_1 \preceq g y_2$ implies that $F(x, y_1) \succeq F(x, y_2),$ for any $x \in X$.

Definition 8. [14] An element $(x,y) \in X\times X$ is called a coupled coincidence point of the mappings $F, G: X\times X \to X$ if $F(x,y) = G(x,y)$ and $F(y,x) = G(y,x)$.

Definition 9. [14] Suppose that $F, G: X\times X \to X$ are two mappings. F is said to be G-increasing with respect to $\preceq$ if for all $x,y,u,v \in X$, with $G(x,y) \preceq G(u,v)$ we have $F(x,y) \preceq F(u,v)$.

Definition 10. [14] Let $F, G: X\times X \to X$. We say that the pair $\{F,G\}$ is generalized compatible if

$$\begin{cases} d\big(F(G(x_n,y_n),G(y_n,x_n)),G(F(x_n,y_n),F(y_n,x_n))\big) \to 0 \\ d\big(F(G(y_n,x_n),G(x_n,y_n)),G(F(y_n,x_n),F(x_n,y_n))\big) \to 0 \end{cases}$$

as $n \to +\infty,$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} G(x_n, y_n) = t_1$$
$$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} G(y_n, x_n) = t_2.$$

Definition 11. [14] Let $F, G: X\times X \to X$ be two maps. We say that the pair $\{F,G\}$ is commuting if

$$F(G(x,y),G(y,x)) = G(F(x,y),F(y,x))$$

for all $x, y \in X$.

As given in [2] Φ denotes the set of all functions $\phi:[0,\infty)\to[0,\infty)$ such that:

1. ϕ is continuous and increasing,
2. $\phi(t) = 0$ if and only if $t = 0,$
3. $\phi(t+s) \le \phi(t) + \phi(s),$ for all $t, s \in [0,+\infty).$

Let Ψ be the set of all the functions $\psi:[0,+\infty)\to[0,+\infty)$ such that $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0^+} \psi(t) = 0.$

Definition 12. [12] A function $\psi:[0,\infty)\to[0,\infty)$ is called an altering distance function if it satisfies the following conditions:

(i) ψ is monotone increasing and continuous;
(ii) $\psi(t) = 0$ if and only if $t = 0.$

In our results in the following section we will use the following class of functions.

$\Psi=\{\psi:[0,\infty)\to[0,\infty)$: an altering distance function$\}$.

$\Phi=\{\varphi:[0,\infty)\to[0,\infty)$: lower semi-continuous functions, $\psi(t)=0$ if and only if $t=0\}$ we have let $\Phi_1=\{\varphi:[0,\infty)\to[0,\infty)$: lower semi-continuous functions, $\varphi(0)\geq 0$, $\varphi(t)>0$, $t>0\}$.

Definition 13. An ultra altering distance function is a continuous, mapping $\varphi:[0,\infty)\to[0,\infty)$ such that $\varphi(0)\geq 0$, $\varphi(t)\neq 0$, $t\neq 0$.

$\Theta=\{\varphi:[0,\infty)\to[0,\infty)$: an ultra altering distance function$\}$.

In Ansari [4] introduced the concept of C-class functions which cover a large class of contractive conditions.

Definition 14. A mapping $f:[0,\infty)^2\to R$ is called C-class function if it is continuous and satisfies following axioms:

(1) $f(s,t)\leq s$

(2) $f(s,t)=s$ implies that either $s=0$ or $t=0$ for all $s,t\in[0,\infty)$.

Note that for some f we have $f(0,0)=0$. We denote C-class functions as C.

MAIN RESULTS

In [14] proved coupled fixed point theorem, we improved this results in the settings of b-metric space.

Theorem 1. Let $(X,\preceq,d)$ be an ordered complete b-metric space (with parameter $s>1$). Assume that $F,G:X\times X\to X$ are two generalized compatible mappings such that F is G - increasing with respect to $\preceq$, G is continuous and has the mixed monotone property, and there exist two elements $x_0,y_0\in X$, with $G(x_0,y_0)\preceq F(x_0,y_0)$ and $G(y_0,x_0)\succeq F(y_0,x_0)$.

Suppose that there exist $\phi\in\Phi$ and $\psi\in\Psi$, f is C-class such that

$$\phi\left(s^a d(F(x,y),F(u,v))\right)\leq f\left(\phi\left(\frac{d(G(x,y),G(u,v))+d(G(y,x),G(v,u))}{2}\right),\ \psi\left(\frac{d(G(x,y),G(u,v))+d(G(y,x),G(v,u))}{2}\right)\right) \quad (1)$$

for all $x,y,u,v\in X$, $a>1$ with $G(x,y)\preceq G(u,v)$. Suppose that for any $x,y\in X$, there exist $u,v\in X$ such that

$$\begin{cases} F(x,y)=G(u,v) \\ F(y,x)=G(v,u). \end{cases} \quad (2)$$

Also suppose that either

(a) F is continuous or

(b) X has the following property

(i) if a non-decreasing sequence $\{x_n\}\to x$, then $x_n\preceq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\}\to y$, for all n.

Then, F and G have a coupled coincidence point in X.

Proof: Let x_0,y_0 be arbitrary point in X such that $G(x_0,y_0)\preceq F(x_0,y_0)$ and $F(y_0,x_0)\preceq G(y_0,x_0)$ (such points exist by hypothesis). From (2), there exists $(x_1,y_1)\in X\times X$ such that $F(x_0,y_0)=G(x_1,y_1)$ and $F(y_0,x_0)=G(y_1,x_1)$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$F(x_n,y_n)=G(x_{n+1},y_{n+1}),\ F(y_n,x_n)=G(y_{n+1},x_{n+1}), \text{ for all } n\in N. \quad (3)$$

First we show that for all $n\in N$, we have

$$G(x_n,y_n)\preceq G(x_{n+1},y_{n+1}) \text{ and } G(y_{n+1},x_{n+1})\preceq G(y_n,x_n). \quad (4)$$

As $G(x_0,y_0)\preceq F(x_0,y_0)$ and $F(y_0,x_0)\preceq G(y_0,x_0)$ as $F(x_0,y_0)=G(x_1,y_1)$ and $F(y_0,x_0)=G(y_1,x_1)$, we have $G(x_0,y_0)\preceq G(x_1,y_1)$ and $G(y_1,x_1)\preceq G(y_0,x_0)$.

Thus (4) holds for $n=0$. Suppose now that (4) holds for some fixed $n\in N$. Since F is G-increasing with respect to $\preceq$, we have

$G(x_{n+1},y_{n+1})=F(x_n,y_n)\preceq$

$F(x_{n+1},y_{n+1})=G(x_{n+2},y_{n+2})$ and

$G(y_{n+2}, x_{n+2}) = F(y_{n+1}, x_{n+1}) \leq$
$F(y_n, x_n) = G(y_{n+1}, x_{n+1})$.

Hence (4) holds for all $n \in N$. For all $n \in N$, denote

$\delta_n = d(G(x_n, y_n), G(x_{n+1}, y_{n+1})) + d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))$.

We can suppose that $\delta_n > 0$ for all $n \in N$. If not, (x_n, y_n) will be a coincidence point and the proof is finished. We claim that for any $n \in N$,
we have $\phi(\delta_{n+1}) \leq \phi(\delta_n)$.
Since $G(x_n, y_n) \leq G(x_{n+1}, y_{n+1})$ and $G(y_n, x_n) \geq G(y_{n+1}, x_{n+1})$, letting $x = x_n$, $y = y_n$,
$u = x_{n+1}$ and $v = y_{n+1}$ in (3) and using (1), we get

$$\begin{aligned}
&\phi\left(s^a d\left(G(x_{n+1}, y_{n+1}), G(x_{n+2}, y_{n+2})\right)\right) \\
&= \phi\left(s^a d\left(F(x_n, y_n), F(x_{n+1}, y_{n+1})\right)\right) \\
&\leq f\left(\phi\left(\frac{d(G(x_n, y_n), G(x_{n+1}, y_{n+1})) + d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))}{2}\right),\right. \\
&\left.\psi\left(\frac{d(G(x_n, y_n), G(x_{n+1}, y_{n+1})) + d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))}{2}\right)\right) \\
&= f\left(\phi\left(\frac{\delta_n}{2}\right), \psi\left(\frac{\delta_n}{2}\right)\right) \qquad (5)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\phi\left(s^a d\left(G(y_{n+2}, x_{n+2}), G(y_{n+1}, x_{n+1})\right)\right) \\
&= \phi\left(s^a d\left(F(y_{n+1}, x_{n+1}), F(y_n, x_n)\right)\right) \\
&\leq f\left(\phi\left(\frac{d(G(y_{n+1}, x_{n+1}), G(y_n, x_n)) + d(G(x_{n+1}, y_{n+1}), G(x_n, y_n))}{2}\right),\right. \\
&\left.\psi\left(\frac{d(G(y_{n+1}, x_{n+1}), G(y_n, x_n)) + d(G(x_{n+1}, y_{n+1}), G(x_n, y_n))}{2}\right)\right) \\
&= f\left(\phi\left(\frac{\delta_n}{2}\right), \psi\left(\frac{\delta_n}{2}\right)\right) \qquad (6)
\end{aligned}$$

Suming (5) and (6), since ϕ is non-decreasing, we obtain that

$$\phi(s^a \delta_{n+1}) \leq f\left(\phi\left(\frac{\delta_n}{2}\right), \psi\left(\frac{\delta_n}{2}\right)\right) \leq \phi\left(\frac{\delta_n}{2}\right) \Rightarrow \delta_{n+1} = \frac{1}{s^a}\delta_n \quad (7)$$

It follows that the sequence (δ_n) is monotone decreasing. Therefore, there is some $\delta \geq 0$ such that $\lim_{n\to\infty} \delta_n = \delta^+$. We shall show that $\delta = 0$. Assume on contrary that $\delta > 0$. Then taking the limit as $n \to +\infty$ (equivalently, $\delta_n \to \delta$) in (7), using the fact that $\lim_{n\to r} \psi(t) > 0$ for all $r > 0$ and ϕ is continuous, we have

$$\phi(s^a \delta) = \lim_{n\to+\infty} \phi(s^a \delta_n) \leq \lim_{n\to+\infty} f\left(\phi\left(\frac{\delta_{n-1}}{2}\right), \psi\left(\frac{\delta_{n-1}}{2}\right)\right)$$

$$= f\left(\phi\left(\frac{\delta}{2}\right), \lim_{\delta_{n-1}\to\delta} \psi\left(\frac{\delta_{n-1}}{2}\right)\right) < \phi(\delta)$$

$$\cdots \frac{1}{s^2} < \frac{1}{s}$$

a contradiction. Thus $\delta = 0$, that is

$$\begin{aligned}
&\lim_{n\to+\infty} \phi\left(d\left(G(x_n, y_n), G(x_{n+1}, y_{n+1})\right)\right. \\
&\left.+ d\left(G(y_n, x_n), G(y_{n+1}, x_{n+1})\right)\right) \\
&= \lim_{n\to\infty} \phi(\delta_n) = 0.
\end{aligned}$$

Hence, by Jovanović et al., (see [11]) (δ_n) i.e., $(d(G(x_n, y_n), G(y_n, x_n)))$ is a b-Cauchy sequence in $X \times X$ endowed with the metric Λ defined by

$$\Lambda((x, y), (u, v)) = d(x, u) + d(y, v)$$

for all $(x, y), (u, v) \in X \times X$. Hence $(G(x_n, y_n))$ and $(G(y_n, x_n))$ are b-Cauchy sequences in (X, d). Now, since (X, d) is b-complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} G(x_n, y_n) = \lim_{n\to+\infty} F(x_n, y_n) = x \quad \text{and}$$
$$\lim_{n\to\infty} G(y_n, x_n) = \lim_{n\to+\infty} F(y_n, x_n) = y. \quad (8)$$

Since the pair $\{F, G\}$ satisfies the generalized compatibility, from (8), we get

$$\lim_{n\to\infty} d\left(F\left(G(x_n, y_n), G(y_n, x_n)\right), G\left(F(x_n, y_n), F(y_n, x_n)\right)\right) = 0 \quad (9)$$

and

$$\lim_{n\to\infty} d\left(F\left(G(y_n, x_n), G(x_n, y_n)\right), G\left(F(y_n, x_n), F(x_n, y_n)\right)\right) = 0$$

Suppose that F is continuous. For all $n \geq 0$, we have

$$\begin{aligned}
&d\left(G(x, y), F\left(G(x_n, y_n), G(y_n, x_n)\right)\right) \\
&\leq s\left(d\left(G(x, y), G\left(F(x_n, y_n), F(y_n, x_n)\right)\right)\right. \\
&\left.+ d\left(G\left(F(x_n, y_n), F(y_n, x_n)\right), F\left(G(x_n, y_n), G(y_n, x_n)\right)\right)\right).
\end{aligned}$$

Taking the limit as $n \to +\infty$, using (8), (9) and the fact that F and G are continuous, we have $G(x,y) = F(x,y)$. Similarly, using (8), (9) and the fact that *F* and *G* are continuous, we have $G(y,x) = F(y,x)$. Thus (x,y) is a coupled coincidence point of *F* and *G*. Now, suppose that (b) holds. By (4) and (8), we have $(G(x_n,y_n))$ is non-decreasing sequence, $G(x_n,y_n) \to x$ and $(G(y_n,x_n))$ is non-decreasing sequence, $G(y_n,x_n) \to y$ as $n \to +\infty$. Thus for all $n \in N$, we have

$$G(x_n,y_n) \le x \quad \text{and} \quad G(y_n,x_n) \ge y. \tag{10}$$

Since the pair $\{F,G\}$ satisfies the generalized compatibility and *G* is continuous, we have

$$\begin{aligned}\lim_{n\to+\infty} G(G(x_n,y_n),G(y_n,x_n)) &= G(x,y)\\ &= \lim_{n\to+\infty} G(F(x_n,y_n),F(y_n,x_n))\\ &= \lim_{n\to+\infty} F(G(x_n,y_n),G(y_n,x_n))\end{aligned} \tag{11}$$

and

$$\begin{aligned}\lim_{n\to+\infty} G(G(y_n,x_n),G(x_n,y_n)) &= G(y,x)\\ &= \lim_{n\to+\infty} G(F(y_n,x_n),F(x_n,y_n))\\ &= \lim_{n\to+\infty} F(G(y_n,x_n),G(x_n,y_n)).\end{aligned} \tag{12}$$

Now, we have

$$\begin{aligned}d(G(x,y),F(x,y)) &\le \lim_{n\to+\infty} d(G(F(x_n,y_n),F(y_n,x_n)),F(x,y))\\ &= \lim_{n\to+\infty} d(F(G(x_n,y_n),G(y_n,x_n)),F(x,y)).\end{aligned}$$

Since *G* has the mixed monotone property, it follows from (10) that

$$G(G(x_n,y_n),G(y_n,x_n)) \le G(x,y) \quad \text{and}$$
$$G(G(y_n,x_n),G(x_n,y_n)) \ge G(y,x).$$

Now using (1), (11) and (12), we get

$$\begin{aligned}&\phi(s^a d(G(x,y),F(x,y))) \le\\ &\lim_{n\to\infty} f\Bigg(\phi\Bigg(\frac{d(G(G(x_n,y_n),G(y_n,x_n)),G(x,y))}{2}\\ &+\frac{d(G(G(y_n,x_n),G(x_n,y_n)),G(y,x))}{2}\Bigg),\\ &\psi\Bigg(\frac{d(G(G(x_n,y_n),G(y_n,x_n)),G(x,y))}{2}\\ &+\frac{d(G(G(y_n,x_n),G(x_n,y_n)),G(y,x))}{2}\Bigg)\Bigg).\end{aligned}$$

Then we obtain that $G(x,y) = F(x,y)$. Similarly, we can show that $G(y,x) = F(y,x)$.

Example 1. Let $f(s,t) = ks$, $X = [0,1]$ endowed with the natural ordered of real numbers. We endow X with the standard metric X

$$d(x,y) = |x-y|^2 \quad \text{for all } x,y \in X.$$

Then (X,d) is a complete b-metric space. Define the mappings $F,G : X \times X \to X$ as follow

$$F(x,y) = \frac{x^3-y^3}{8}, \quad \text{if } x \ge y,$$
$$F(x,y) = 0, \quad \text{if } x < y,$$

and

$$G(x,y) = x^3 - y^3, \quad \text{if } x \ge y$$
$$G(x,y) = 0, \quad \text{if } x < y.$$

First we prove that F is G-increasing. Let $(x,y),(u,v) \in X \times X$ with $G(x,y) \le G(u,v)$. We consider the following cases.

Case 1: If $x < y$ then $F(x,y) = 0 \le F(u,v)$.

Case 2: If $x \ge y$, if $u \ge v$, then

$$\begin{aligned}&G(x,y) \le G(u,v)\\ &\Rightarrow x^3 - y^3 \le u^3 - v^3\\ &\Rightarrow \frac{x^3-y^3}{8} \le \frac{u^3-v^3}{8}\\ &\Rightarrow F(x,y) \le F(u,v).\end{aligned}$$

If $u < v$, then

$$\begin{aligned}&G(x,y) \le G(u,v)\\ &\Rightarrow 0 \le x^3 - y^3 \le 0\\ &\Rightarrow x^3 = y^3\\ &\Rightarrow F(x,y) = 0 \le F(u,v).\end{aligned}$$

Thus we have F is G - increasing.

Now we prove that for any $x, y \in X$, these exist $u, v \in X$ such that

$$\begin{cases} F(x, y) = G(u, v) \\ F(y, x) = G(v, u). \end{cases}$$

Let $(x, y) \in X \times X$ be fixed. We consider the following cases.

Case 1: If $x = y$, then we have, $F(x, y) = 0 = G(x, y)$ and $F(y, x) = 0 = G(y, x)$.

Case 2: If $x > y$, then we have

$$F(x, y) = \frac{x^3 - y^3}{8} = G\left(\frac{x}{2}, \frac{y}{2}\right)$$

and $F(y, x) = 0 = G\left(\frac{y}{2}, \frac{x}{2}\right)$.

Case 3: If $x < y$, then we have

$$F(x, y) = 0 = G\left(\frac{x}{2}, \frac{y}{2}\right)$$

and $F(y, x) = \frac{y^3 - x^3}{8} = G\left(\frac{y}{2}, \frac{x}{2}\right)$.

Now we prove that G is continuous and has the mixed monotone property. Clearly G is continuous. Let $(x, y) \in X \times X$ be fixed. Suppose that $x_1, x_2 \in X$ are such that $x_1 < x_2$. We discuss the following cases.

Case 1: If $x_1 < y$, then we have,

$$G(x_1, y) = 0 \le G(x_2, y).$$

Case 2: If $x_2 > x_1 > y$, then we have

$$G(x_1, y) = x_1^3 - y^3 \le x_2^3 - y^3 = G(x_2, y).$$

Similarly, we can show that if $y_1, y_2 \in X$ are such that $y_1 < y_2$, then $G(x, y_1) \ge G(x, y_2)$.

Now, we prove that the pair $\{F, G\}$ satisfies the generalized compatibility hypothesis. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that

$$t_1 = \lim_{n\to+\infty} F(x_n, y_n) = \lim_{n\to+\infty} G(x_n, y_n)$$

and $t_2 = \lim_{n\to+\infty} F(y_n, x_n) = \lim_{n\to+\infty} G(y_n, x_n)$.

Then it must be that $t_1 = t_2 = 0$ and one can easily prove that

$$\begin{cases} \lim_{n\to+\infty} d\big(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n))\big) = 0 \\ \lim_{n\to+\infty} d\big(F(G(y_n, x_n), G(x_n, y_n)), G(F(y_n, x_n), F(x_n, y_n))\big) = 0. \end{cases}$$

Now we prove that there exist two elements $x_0, y_0 \in X$ with $G(x_0, y_0) \le F(x_0, y_0)$ and $G(y_0, x_0) \ge F(y_0, x_0)$.

Since we have $G\left(0, \frac{1}{2}\right) = 0 = F\left(0, \frac{1}{2}\right)$ and $G\left(\frac{1}{2}, 0\right) = \frac{1}{8} \ge \frac{1}{64} = F\left(\frac{1}{2}, 0\right)$.

Now, let $\phi, \psi : [0, \infty) \to [0, \infty)$ be defined as $\phi(t) = t$, for all $t \in [0, \infty)$, clearly $\phi \in \Phi$. We next verify the contraction (1) for all $x, y, u, v \in X$, with $G(x, y) \le G(u, v)$ and $G(v, u) \le G(y, x)$. We have

$$\phi\big(2^4(d(F(x, y), F(u, v)))\big) = 16\big(d(F(x, y), F(u, v))\big)$$

$$= 16|F(x, y) - F(u, v)|^2$$

$$= \frac{16}{64}|G(x, y) - G(u, v)|^2$$

$$\le \frac{1}{4}\left[|G(x, y) - G(u, v)|^2 + |G(y, x) - G(v, u)|^2\right]$$

$$= \frac{1}{2}\left[\phi\left(\frac{d(G(x, y), G(u, v)) + d(G(y, x), G(v, u))}{2}\right)\right]$$

$$= k\left[\phi\left(\frac{d(G(x, y), G(u, v)) + d(G(y, x), G(v, u))}{2}\right)\right]$$

$$= f\left\{\phi\left(\frac{d(G(x, y), G(u, v)) + d(G(y, x), G(v, u))}{2}\right), \psi\left(\frac{d(G(x, y), G(u, v)) + d(G(y, x), G(v, u))}{2}\right)\right\}.$$

Where $k = \frac{1}{2}$, hence condition (1) is satisfied.

Thus all the requirements of Theorem 1 are satisfied and $(0,0)$ is coupled coincidence point of F and G.

CONCLUSION

In this paper, by using the C-class functions and a new approach, we present some coupled coincidence point result for a pair of mappings satisfying generalized (ϕ, ψ)-weakly contractive conditions in the setting of ordered b-metric spaces. An application and example are given in [15].

REFERENCE

[1] Arshad, M., Karapinar, E., Ahmad, J. Some unique fixed point theorem for rational contractions in partially ordered metric spaces. J. Inequal.Appl. 2013, (2013).

[2] Alotaibi, A., Alsulami, S.M. Coupled coincidence points for monotone operators in partially ordered metric spaces. Fixed Point Theory Appl. 2011, (2011).

[3] Aghajani, A., Abbas, M., Roshan, J.R. Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces. Math. Slov. 4:941960, (2014).

[4] Ansari, A.H., Chandok, S., Ionescu, C. Fixed point theorems on b-metric spaces for weak contractions with auxiliary functions. J. Inequal. Appl. 2014:429, (2014).

[5] Bhaskar, T. G., Lakshmikantham, V. Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. 65,1379-1393, (2006).

[6] Boriceanu, M., Bota, M., Petrusel, A. Multivalued fractals in b-metric spaces. Cent. Eur. J. Math. 8(2):367277, (2010).

[7] Ćirić, L. B., Cakić, N., Rajović, M., Ume, J.S. Monotone generalized nonlinear contractions in partially ordered metric spaces. Fixed Point Theory Appl. 2008, (2008).

[8] Czerwik, S. Nonlinear set-valued contraction mappings in b-metric spaces. Atti. Sem. Mat. Fis. Univ. Modena, 46(2):263276, (1998).

[9] Guo, D., Lakshmikantham, V. Coupled fixed points of nonlinear operators with applications. Nonlinear Anal. 11(5), 623-632, (1987).

[10] Huang, H., Paunović, Lj., Radenović, S. On Some new fixed point results for rational Geraghty contractive mappings in ordered b- metric spaces. J. Nonlinear Sci. Appl. 8, 800-807, (2015).

[11] Jovanović, M., Kadelburg, Z., Radenović, S. Common Fixed Point Results in Metric Type Spaces. Fixed point theory and applications, vol. 2010, Article ID978121, (2010). doi: 10.1155/2010/978121.

[12] Khan, M. S., Swaleh, M., Sessa, S. Fixed point theorems by altering distances between the points. Bulletin of the Australian Mathematical Society, 30, 1-9, (1984).

[13] Lakshmikantham, V., Ćirić, L. B.Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. 70, 4341-4349, (2009).

[14] Nawab, H., Mujahid, A., Akbar, A. Jamshaid, A. Coupled coincidence point results for a generalized compatible pair with applications. Fixed Point Theory and Applications, 2014:62, (2014).

[15] Paunović, Lj. Resolving systems of nonlinear integral equations via C-class functions. UNIVERSITY THOUGHT, Publication in Natural Sciences, Vol. 7, No.2, (2017). doi:10.5937/univtho7-14709.

[16] Paunović, Lj., Kaushik, P., Kumar. S. Some applications with new admissibility contractions in b-metric spaces. J. Nonlinear Sci. Appl. 10(2017),4162-4174, (2017). doi:10.22436/jnsa.010.08.12

[17] Paunović, Lj., Sangurlu, M. Ansari, A. H. Fixed point results for weak S-contractions via C-class functions. UNIVERSITY THOUGHT, Publication in Natural Sciences, Vol. 9, No. 2, (2019). https://doi.org/10.5937/univtho9-21156

[18] Wu, Y. New fixed point theorems and applications of mixed monotone operator. J. Math. Anal. Appl. 341, 883-893, (2008).